

Surfaces associated with theta function solutions of the periodic 2D-Toda lattice

A. M. Grundland^{#†1}, M. Y. Mo[#]

[#] Centre de recherche mathématiques, Université de Montréal, C. P. 6128,
Succursale centre-ville, Montréal (Quebec) H3C 3J7, Canada

[†] Université du Québec, Trois-Rivières CP500 (QC) G9A 5H7, Canada

Keywords: Surface immersions, Toda lattice

Abstract

The objective of this paper is to present some geometric aspects of surfaces associated with theta function solutions of the periodic 2D-Toda lattice. For this purpose we identify the $(N^2 - 1)$ -dimensional Euclidean space with the $\mathfrak{su}(N)$ algebra which allows us to construct the generalized Weierstrass formula for immersion for such surfaces. The elements characterizing surface like its moving frame, the Gauss-Weingarten and the Gauss-Codazzi-Ricci equations, the Gaussian curvature, the mean curvature vector and the Wilmore functional of a surface are expressed explicitly in terms of any theta function solution of the Toda lattice model. We have shown that these surfaces are all mapped into subsets of a hypersphere in \mathbb{R}^{N^2-1} . A detailed implementations of the obtained results are presented for surfaces immersed in the $\mathfrak{su}(2)$ algebra and we show that different Toda lattice data correspond to different subsets of a sphere in \mathbb{R}^3 .

1 Introduction

Modern surface theory is a subject that has generated a great deal of interests and activities in several branches of mathematics as well as in various areas of physical science. In particular, surfaces associated to \mathbb{CP}^N sigma models provide us with a rich class of geometric objects (See, e.g. Helein [17], [18]). Until very recently, the immersion of 2-dimensional surfaces obtained through the Weierstrass formula for these models has been known only in low dimensional Euclidean spaces. The expressions describing locally minimal surfaces (i.e. has zero mean curvature) immersed in 3-dimensional Euclidean space were first

¹This work was supported in part by research grants from NSERC of Canada. M. Y. Mo acknowledges the postdoctoral fellowship awarded by the laboratory of mathematical physics of the CRM, université de Montreal.

derived one and a half century ago by A. Enneper [14], and K. Weierstrass [37]. It starts by introducing a pair of holomorphic functions ψ_1 and ψ_2 and then one can introduce the 3-component real-valued vector $X(\xi, \bar{\xi})$ as follows

$$X(\xi, \bar{\xi}) = \mathbf{Re} \left(i \int_{\gamma} (\psi_1^2 + \psi_2^2) d\xi^1, \int_{\gamma} (\psi_1^2 - \psi_2^2) d\xi^1, -2 \int_{\gamma} \psi_1 \psi_2 d\xi^1 \right) \quad (1.1)$$

where γ is an arbitrary curve in \mathbb{C} . The parametric lines $\mathbf{Re}\xi = \text{const.}$ and $\mathbf{Im}\xi = \text{const.}$ on these type of surfaces are the minimal lines. This expression proved to be the most powerful tool for construction and investigation of this type of surfaces in \mathbb{R}^3 (for a review of the subject, see e.g. Kenmotsu [22]).

An extension of the Weierstrass representation (1.1) for surfaces immersed into multi-dimensional Riemannian spaces in connection with linear systems (e.g. multi-dimensional Dirac equations) has been developed over the last decades by many authors (e.g. [5], [11], [15], [24]). The main goal is to provide a self-contained, comprehensive approach to the Weierstrass formulae for immersions. For this purpose, it is convenient to formulate equations defining the immersion directly in the matrix form which take their values in a finite dimensional Lie algebra. The main advantages of this procedure appear when group analysis of the immersion makes it possible to construct regular algorithms for finding certain classes of surfaces without referring to any additional considerations. Instead, we proceed directly from the given model of equations. The task of finding an increasing number of surfaces is facilitated by the group properties of the model considered. The rich character of this formulation makes the immersion formula a rather interesting object to investigate.

This paper is concerned with some geometric aspects of smooth 2-dimensional surfaces arising from the study of theta function solutions of the 2D Toda lattice model. To achieve this we identify \mathbb{R}^{N^2-1} with the Lie algebra $\mathfrak{su}(N)$ which allows us to construct the generalized Weierstrass formula for immersion of surfaces. This formula is a consequence of conservation laws of the model. We show that there exist $N^2 - 1$ real-valued functions which can be treated as coordinates of the surface immersed in the $\mathfrak{su}(N)$ algebra. We reformulate explicitly the structural equations for the immersion in Cartan's language of moving frames on a surface. These are expressed in terms of any theta function solution of the Toda model. In fact, this approach allows us to find explicitly the form of the Gauss-Weingarten and the Gauss-Codazzi-Ricci equations in terms of this type of solution. The first and second fundamental forms, the Gauss curvature and the mean curvature vector are the central concepts described here. We present in details an implementation of these results for surfaces immersed in the $\mathfrak{su}(2)$ algebra. In the case of $\mathfrak{su}(2)$ we show that different Toda lattice data correspond to different subsets of a sphere in \mathbb{R}^3 .

From the point of view of the 2D-Toda lattice model, this paper addresses the following two questions.

1. Starting from the theta function of the 2D-Toda lattice model one can derive the conservation laws and immersion of a 2D-surface in multi-dimensional space $\mathbb{R}^{N^2-1} = \mathfrak{su}(N)$. What are the properties of this surface?

2. Given that there is a connection between 2D-Toda lattice models and surfaces immersed in $\mathfrak{su}(N)$, do these surfaces tell us anything new? Is this connection useful?

Finally, let us note that the outlined approach to the study of surfaces leads to the following potential applications of the theory of surfaces associated to the doubly periodic solutions of the 2D-Toda lattice.

In physics the doubly periodic solutions have found applications in such varied areas as field theory, quantum field theory and string theory [1], [20], [34], statistical physics [7], [28], phase transitions (e.g. growth of crystals, deformations of membranes, dynamics of vortex sheets, surfaces waves etc. [6]), fluid dynamics, e.g. the motion of boundaries between regions of different densities and viscosities [21].

In biochemistry and biology, surfaces have been shown to play an essential role in several applications to nonlinear phenomena in the study of biological membranes and vesicles, for example long protein molecules [26], [8], [30], the Canham-Helfrich membrane models [25]. These macroscopic models can be derived from microscopic ones and allow us to explain basic features and equilibrium shapes both for biological membranes and for liquid interfaces [9], [36]. In chemistry there are applications e.g. to energy and momentum transport along a polymer molecule [8].

The results in this paper give a systematic way of constructing surfaces from doubly periodic solutions which could be applied to the above areas of research.

In mathematics, the analysis described in this paper could be extended to a systematic description of surfaces in the Lie algebras/ Lie groups framework and isomonodromic deformations in connection with surfaces (e.g. [2], [3] and the references therein).

The paper is organized as follows. In section 2, we give a brief introduction to the standard techniques in integrable systems and in particular the 2D-Toda lattice. In section 3, we explained how to construct solutions of the 2D-Toda lattice from theta functions of Riemann surfaces, while section 4 defines immersions from these solutions of the 2D-Toda lattice and introduced some basic results in surface immersion theory. In section 5 we expressed these immersions in terms of theta functions and derived the moving frame, the first and second fundamental forms and the mean curvature vector. Finally, section 6 sees the application of these results to the $\mathfrak{su}(2)$ case in which the surfaces obtained have constant mean and Gaussian curvatures.

2 Integrable systems structure of the 2D-Toda lattice

The periodic Toda lattice is equivalent to the following compatibility equations

$$\bar{\partial}A - \partial B = [A, B], \quad B = -A^\dagger, \quad (2.1)$$

$$A = \begin{pmatrix} \partial u_0 & 0 & 0 & \dots & 0 & U_{0,N} \\ U_{1,0} & \partial u_1 & 0 & \dots & 0 & 0 \\ 0 & U_{2,1} & \partial u_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \partial u_{N-1} & 0 \\ 0 & 0 & 0 & \dots & U_{N,N-1} & \partial u_N \end{pmatrix} \quad (2.2)$$

where $\partial = \partial_\xi$, $\bar{\partial} = \partial_{\bar{\xi}}$, u_i are real-valued functions $u_i : \mathbb{C} \rightarrow \mathbb{R}$ such that $\sum_{i=1}^N u_i = 0$ and $U_{i,j} = e^{u_i - u_j}$. Therefore A is in the $\mathfrak{sl}(n, \mathbb{C})$ algebra.

To understand the integrable systems structure, first let us write the zero curvature condition as one that involves a complex parameter λ

$$\bar{\partial} A_\lambda - \partial B_\lambda = [A_\lambda, B_\lambda], \quad B_\lambda = -\rho(A_\lambda), \quad \lambda \in \mathbb{C}$$

$$A_\lambda = \begin{pmatrix} \partial u_0 & 0 & 0 & \dots & 0 & \lambda U_{0,N} \\ U_{1,0} & \partial u_1 & 0 & \dots & 0 & 0 \\ 0 & U_{2,1} & \partial u_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \partial u_{N-1} & 0 \\ 0 & 0 & 0 & \dots & U_{N,N-1} & \partial u_N \end{pmatrix}$$

where ρ is the involution on the series of matrix defined by

$$\rho \left(\sum_{-\infty}^{\infty} X_i \lambda^i \right) = \sum_{-\infty}^{\infty} X_i^\dagger \lambda^{-i}, \quad X_i \in \mathfrak{sl}(n, \mathbb{C}) \quad (2.3)$$

We can represent A_λ as follows

$$A_\lambda = A_0 + A_1 \Lambda, \quad B = -A_0^\dagger - A_1^\dagger \Lambda^{-1}$$

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \lambda \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad \Lambda^{-1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 \\ \lambda^{-1} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

where A_0, A_1 are diagonal matrices.

One can now look for formal solutions F of the linear equations

$$\partial F = -A_\lambda F, \quad \bar{\partial} F = -B_\lambda F \quad (2.4)$$

near $\lambda = 0$ and $\lambda = \infty$ respectively. Since $U_{i,j} = e^{u_i - u_j}$, the determinant of A_1 equals 1. It is a standard result due to Drinfeld and Sokolov [12] that near $\lambda = \infty$, there exists a formal solution F_∞ of the form

$$F_\infty = T_\infty \exp(\xi \Lambda) \quad (2.5)$$

where T_∞ is a formal power series in Λ^{-1} of the form

$$T_\infty = D_0 + D_1 \Lambda^{-1} + \dots \quad (2.6)$$

with diagonal matrices D_i such that D_0 is invertible. Similarly, near $\lambda = 0$, there exists a formal solution F_0 of the form

$$F_0 = T_0 \exp(\bar{\xi} \Lambda^{-1})$$

with similar properties. The matrices T_0 and T_∞ are called the dressing matrices.

Let L_i and \bar{L}_i denote the following matrices,

$$L_i = (T_\infty \Lambda^i T_\infty^{-1})_+, \quad \bar{L}_i = (T_0 \Lambda^{-i} T_0^{-1})_-, \quad i = 1, \dots, \infty \quad (2.7)$$

where $+$, $-$ denotes the polynomial parts in Λ and Λ^{-1} respectively. From now on, we will suppress the λ dependence of L_i .

Usual argument in integrable systems theory shows that one can then define infinitely many commuting flows according to (See e.g. [12])

$$\begin{aligned} \partial_{t_i} A_\lambda - \partial L_i &= [A_\lambda, L_i], & \partial_{\bar{t}_i} A_\lambda - \partial \bar{L}_i &= [A_\lambda, \bar{L}_i] \\ \partial_{t_i} B_\lambda - \bar{\partial} L_i &= [B_\lambda, L_i], & \partial_{\bar{t}_i} B_\lambda - \bar{\partial} \bar{L}_i &= [B_\lambda, \bar{L}_i] \end{aligned} \quad (2.8)$$

To see how this works, consider, for example, the case of L_i . First make a gauge transformation

$$\begin{aligned} \partial + \hat{A}_\lambda &= \mathbb{G}^{-1}(\partial + A_\lambda)\mathbb{G} \\ \mathbb{G} &= \text{diag}(e^{u_0}, e^{u_1}, \dots, e^{u_N}) \end{aligned}$$

so that \hat{A}_λ has Λ as its leading term. That is

$$\hat{A}_\lambda = \Lambda + \hat{A}_0$$

we have

$$[\partial + \hat{A}_\lambda, L_i] = [\partial + \hat{A}_\lambda, (\mathbb{G}^{-1} T_\infty \Lambda^i T_\infty^{-1} \mathbb{G}) - (\mathbb{G}^{-1} T_\infty \Lambda^i T_\infty^{-1} \mathbb{G})_{<0}] \quad (2.9)$$

where $(\mathbb{G}^{-1} T_\infty \Lambda^i T_\infty^{-1} \mathbb{G})_{<0}$ is a series in negative powers of Λ only. That is, it has no constant term in Λ .

The series T_∞ is the gauge that transforms $\partial + A_\lambda$ to $\partial + \Lambda$, that is,

$$T_\infty(\partial + \Lambda)T_\infty^{-1} = \partial + A_\lambda$$

we have

$$[\partial + \hat{A}_\lambda, \mathbb{G}^{-1} T_\infty \Lambda^i T_\infty^{-1} \mathbb{G}] = \mathbb{G}^{-1} T_\infty [\partial + \Lambda, \Lambda^i] T_\infty^{-1} \mathbb{G} = 0,$$

therefore we have

$$[\partial + \hat{A}_\lambda, L_i] = [\partial + \hat{A}_\lambda, -(\mathbb{G}^{-1} T_\infty \Lambda^i T_\infty^{-1} \mathbb{G})_{<0}] \quad (2.10)$$

The left hand side is a positive series in Λ while the right hand side is a non-positive series in Λ , we see that it must be a constant in Λ . According to the form of the series expansion (2.6), both sides of (2.10) must be a diagonal matrix. We can therefore set this diagonal matrix to be $\partial_{t_i} \hat{A}_\lambda$. By applying the gauge transformation \mathbb{G}^{-1} to these equations, we obtain the equations (2.8).

The argument for L_i and B is similar, but we use the following instead of (2.9)

$$\begin{aligned}\bar{\partial} F_\infty &= -B_\lambda F_\infty \\ \bar{\partial} T_\infty \exp(\xi \Lambda) &= -B_\lambda T_\infty \exp(\xi \Lambda) \\ \bar{\partial} T_\infty &= -B_\lambda T_\infty\end{aligned}$$

All these flows commute with each other and each flow defines an invariant of the 2D-Toda lattice.

3 Theta function solutions of the Toda lattice

In this section we present some basic facts concerning multi-dimensional theta functions on Riemann surfaces that will provide the tool to construct explicit solutions of the Toda lattice. Later we will see how to construct surfaces from these solutions.

3.1 Basic results of theta function

Let us first remind ourselves some useful facts of theta function that will allow us to construct explicitly meromorphic functions and functions with essential singularities of exponential type. For a review of the subject see e.g. [27].

First choose a canonical basis of cycles $\{a_i, b_i\}$ on a Riemann surface Σ , and let ω_i be 1-forms dual to this basis, that is

$$\int_{a_i} \omega_j = \delta_{ij}, \quad \int_{b_i} \omega_j = \tau_{ij}$$

We can define a lattice $L(M)$ in \mathbb{C}^g by using the columns of the $g \times 2g$ matrix (I_d, Π) , where I_d is the $g \times g$ identity matrix and $(\Pi)_{ij} = \tau_{ij}$. The torus $\mathbb{C}^g \backslash L(M)$ is called the Jacobian $Jac(\Sigma)$ of the Riemann surface.

The theta function associated to the Riemann surface is a function $\Theta : \mathbb{C}^g \rightarrow \mathbb{C}$ defined by

$$\Theta(\vec{z}) = \sum_{\vec{n} \in \mathbb{Z}^g} e^{i\pi \vec{n} \cdot \Pi \vec{n} - 2i\pi \vec{z} \cdot \vec{n}}$$

The theta function has the following periodicity.

Proposition 3.1 *Let e^k, τ^k be the columns of the matrices I_d and Π respectively, then*

$$\begin{aligned}\Theta(\vec{z} + e^k) &= \Theta(\vec{z}) \\ \Theta(\vec{z} + \tau^k) &= \exp 2\pi i \left(-z_k - \frac{\tau_{kk}}{2} \right) \Theta(\vec{z})\end{aligned}\quad (3.1)$$

Let $U : \Sigma \rightarrow \text{Jac}(\Sigma)$ be the Abel map, then the theta function composite with the Abel map has g zeros on Σ . In fact, let $D = \sum_{i=1}^g \gamma_i$ be a divisor of degree g on Σ , then the following multi-valued function

$$\Theta(U(p) - U(D) - K)$$

has zeros at the g points γ_i , where the vector $K = (K_1, \dots, K_g)$ is the Riemann constant

$$K_j = \frac{2\pi i + \tau_{jj}}{2} - \frac{1}{2\pi i} \sum_{l \neq j} \int_{a_l} (\omega_l(P) \int_{P_0}^P \omega_j)$$

3.2 Baker functions

We will now define the Baker functions, which would serve as the fundamental solutions of the linear system (2.4).

Definition 3.1 *Let Σ be a Riemann surface of genus g , and Q_1, \dots, Q_l are l points on Σ . Let k_i^{-1} be local coordinates in neighborhoods of these points ($k_i(Q_i) = \infty$), and D be a divisor on $\Sigma/(Q_1 \cup \dots \cup Q_l)$. Then a Baker l point function $f(P)$ is a meromorphic function on $\Sigma/(Q_1 \cup \dots \cup Q_l)$ that has pole divisor D and that near Q_i , $f \exp(-q_i(k_i))$ is holomorphic for some polynomial q_i in k_i .*

A useful theorem that will be used throughout the construction is the Riemann-Roch theorem.

Theorem 3.2 (Riemann-Roch) *Let D be the pole divisor of the Baker l point function $f(P)$ as in definition 3.1 and let d be its degree. If D is not a special divisor, then the Baker l point function that has pole divisor D forms a linear space that has dimension $\max(d - g + 1, 0)$, where d is the degree of D . (See, e.g. [13]).*

In particular, if the divisor D is of degree g , then the Baker l point function with pole divisor D will be unique up to a constant factor. If the divisor D is of degree $g + n$, but n zeros of the Baker l point function are given, then the Baker l point function can also be uniquely determined up to a constant factor. We therefore have the following

Corollary 3.3 *Let D be a non-special divisor of degree g , and Q_1, Q_2 be two fix points on a Riemann surface Σ of genus g , then there exist unique meromorphic functions f_n on $\Sigma/(Q_1 \cup Q_2)$ that have pole divisor D and such that*

$$\begin{aligned} f_n(P) &= k_1^n \left(\sum_{i=0}^{\infty} h_{n,i}^{(1)} k_1^{-i} \right) \exp(k_1 \xi), \quad P \rightarrow Q_1, \\ f_n(P) &= k_2^{-n} \left(1 + \sum_{i=1}^{\infty} h_{n,i}^{(2)} k_2^{-i} \right) \exp(k_2 \bar{\xi}), \quad P \rightarrow Q_2, \end{aligned} \quad (3.2)$$

where k_i^{-1} are local coordinates near Q_i , that is, $k_i^{-1}(Q_i) = 0$.

In fact, the functions f_n can be expressed in terms of the theta function on Σ .

Let Ω^1 and Ω^2 be the normalized meromorphic 1-forms on Σ (that is, $\int_{a_j} \Omega^i = 0$) that are holomorphic on Σ/Q_1 and Σ/Q_2 respectively, and that

$$\begin{aligned} \Omega^1 &= dk_1 + O(k_1^{-1}), \quad P \rightarrow Q_1, \\ \Omega^2 &= dk_2 + O(k_2^{-1}), \quad P \rightarrow Q_2 \end{aligned}$$

where $O(k^{-1})$ are terms of order k^{-1} . Let us denote the b -periods of these 1-forms by the vectors $\mathbb{B}_1, \mathbb{B}_2$.

We can now express the functions f_n in terms of theta functions.

Proposition 3.4 *The functions f_n can be expressed in terms of theta functions as follows*

$$\begin{aligned} f_n(P) &= C \frac{\Theta(U(P) - U(D) + n(U(Q_2) - U(Q_1)) + \xi \mathbb{B}_1 + \bar{\xi} \mathbb{B}_2 - K)}{\Theta(U(P) - U(D) - K)} \\ &\times \left(\frac{\Theta(U(P) - U(Q_2) - e)}{\Theta(U(P) - U(Q_1) - e)} \right)^n \exp \left(\xi \int^P \Omega^1 + \bar{\xi} \int^P \Omega^2 \right) \end{aligned} \quad (3.3)$$

where Θ is the theta function and e is a vector of the form

$$e = U(x_1) + \dots + U(x_{g-1}) + K$$

where x_i are $g-1$ arbitrary points on Σ and C is a normalization constant.

Proof. Since the factors $\Theta(U(P) - U(Q_1) - e)$ and $\Theta(U(P) - U(Q_2) - e)$ has zeros at $\{x_i, Q_1\}$ and $\{x_i, Q_2\}$ respectively, the factor $\left(\frac{\Theta(U(P) - U(Q_2) - e)}{\Theta(U(P) - U(Q_1) - e)} \right)^n$ on the right hand side of (3.3) has an order n zero at Q_2 and an order n pole at Q_1 . Similarly, the factor $\Theta(U(P) - U(D) - K)$ has zeros at the points in D . Therefore the right hand side of (3.3) has poles at D with the asymptotic form indicated as in (3.2). One can now verify that the right hand side of (3.3) is in fact single-valued by using the periodicity of the theta function (3.1). \square

To obtain solutions of the Toda lattice from this, we need the following lemma [13]

Lemma 3.5 *Suppose that there is an involution $\rho : \Sigma \rightarrow \Sigma$ that permutes Q_1 and Q_2 such that $k_1 = -\overline{\rho(k_2)}$. If the divisor D that defines f_n in corollary 3.3 is such that $D + \rho(D)$ is the zero divisor of a differential ω of third kind with simple poles at Q_1 and Q_2 , then the coefficients $h_{n,0}^{(1)}$ are real.*

Proof. By a differential of third kind, we mean that ω has only simple poles at Q_1 and Q_2 , with residues 1 and -1 respectively.

Since $D + \rho(D)$ is the zero divisor of ω , the following differential

$$\tilde{\omega} = f_n(P)\overline{f_n}(\rho(P))\omega$$

is again a differential of third kind with simple poles only at Q_1 and Q_2 . Its residues are $(-1)^n h_{n,0}^{(1)}$ and $-(-1)^n \overline{h_{n,0}^{(1)}}$ respectively. Since the sum of residues is zero, the lemma follows. \square

If we now let f_n be functions on a Riemann surface defined by a polynomial

$$y^{N+1} + a_0 \lambda^m + \overline{a_0} \lambda^{-m} + P(y, \lambda) = 0 \quad (3.4)$$

such that $m, N+1$ are relatively prime and that

$$\overline{P}(\overline{y}, \overline{\lambda}^{-1}) = P(y, \lambda)$$

where $P(y, \lambda)$ is a polynomial in y and λ . We can define an the involution ρ on this Riemann surface by

$$\rho : (y, \lambda) \mapsto (\overline{y}, \overline{\lambda}^{-1}) \quad (3.5)$$

Since the Riemann surface is branched at $Q_1 = (y = \infty, \lambda = \infty)$ and $Q_2 = (y = \infty, \lambda = 0)$, we can choose local coordinates k_1 and k_2 to be

$$k_1 = \lambda^{\frac{1}{N+1}}, \quad k_2 = -\lambda^{-\frac{1}{N+1}} \quad (3.6)$$

We can now state the main theorem of this section

Theorem 3.6 *Suppose the f_n in corollary 3.3 defined by a divisor D satisfies the condition in lemma 3.5 with local coordinates and involutions defined by (3.6) and (3.5) respectively. Then if $h_{n,0}^{(1)} h_{n+1,0}^{(1)} < 0$, the functions f_n satisfy the following*

$$\begin{aligned} \partial f_n &= -U_{n,n-1}^2 f_{n-1} \\ \overline{\partial} f_n &= f_{n+1} + 2\overline{\partial} u_n f_n, \quad e^{2u_n} = -h_{n,0}^{(1)} \end{aligned} \quad (3.7)$$

Proof. We shall now denote Q_1 by Q_∞ and Q_2 by Q_0 and change all the indices accordingly.

To prove the first equation in (3.7), recall that the functions f_n has the following asymptotic behavior

$$\begin{aligned} f_n(P) &= k_\infty^{-n} \left(\sum_{i=0}^{\infty} h_{n,i}^{(\infty)} k_1^{-i} \right) \exp(k_\infty \xi), \quad P \rightarrow Q_\infty, \\ f_n(P) &= k_0^n \left(1 + \sum_{i=1}^{\infty} h_{n,i}^{(0)} k_2^{-i} \right) \exp(k_0 \bar{\xi}), \quad P \rightarrow Q_0, \end{aligned}$$

Suppose that $h_{n,0}^{(1)}$ are negative and let $e^{2u_n} = -h_{n,0}^{(1)}$, then

$$\partial f_n + U_{n,n-1}^2 f_{n-1} = k_\infty^{-n} O(1) \exp(k_\infty \xi), \quad P \rightarrow Q_\infty$$

where $O(1)$ is a term that is holomorphic at Q_∞ , and

$$\partial f_n + U_{n,n-1}^2 f_{n-1} = k_0^{n-1} O(1) \exp(k_0 \bar{\xi}), \quad P \rightarrow Q_0$$

where $O(1)$ is a term that is holomorphic at Q_0 . We see that the left hand side is a function R with poles only at D and same asymptotic behavior as f_{n-1} except that near Q_∞ , it has a zero of order n while f_{n-1} has a zero of order $n-1$. Therefore $\frac{R}{f_{n-1}} = 0$ at Q_∞ . Now by the Riemann-Roch theorem (theorem 3.2), we see that $R = 0$.

We now prove the second equation in (3.7). We see that

$$\bar{\partial} f_n - f_{n+1} - 2\bar{\partial} u_n f_n = k_0^n O(1) \exp(k_0 \bar{\xi}), \quad P \rightarrow Q_0$$

while near Q_∞ , it has the following asymptotics

$$\bar{\partial} f_n - f_{n+1} - 2\bar{\partial} u_n f_n = k_0^{n+1} O(1) \exp(k_\infty \xi), \quad P \rightarrow Q_\infty$$

Since $h_{n,0}^{(1)} = -e^{2u_n}$. Then by similar argument as before, we see that the both sides must be zero. \square

We can now express the solutions in terms of theta functions.

Proposition 3.7 *The functions u_i that solve the Toda lattice equation (2.1) can be expressed in terms of theta function as*

$$\begin{aligned} u_n &= \frac{1}{2} \log \left| \frac{\Theta(U(Q_\infty) - U(D) + n(U(Q_0) - U(Q_\infty)) + \xi \mathbb{B}_1 + \bar{\xi} \mathbb{B}_2 - K)}{\Theta(U(Q_0) - U(D) + n(U(Q_0) - U(Q_\infty)) + \xi \mathbb{B}_1 + \bar{\xi} \mathbb{B}_2 - K)} \right| \\ &+ c + c_n n \end{aligned} \tag{3.8}$$

where c and c_n are constants.

Proof. We first show that the set of equations (3.7) implies (2.1). To see this, note that the condition

$$\bar{\partial} A_\lambda - \partial B_\lambda = [A_\lambda, B_\lambda]$$

is equivalent to

$$[\partial + A_\lambda, \bar{\partial} + B_\lambda] = 0$$

We can make a gauge transformation $\partial + A_\lambda \mapsto \mathbb{G}(\partial + A_\lambda)\mathbb{G}^{-1}$, $\partial + B_\lambda \mapsto \mathbb{G}(\partial + B_\lambda)\mathbb{G}^{-1}$ by the gauge

$$\mathbb{G} = \text{diag}(e^{u_0}, e^{u_1}, \dots, e^{u_N})$$

This will change the Lax pair into the following

$$\begin{aligned} \partial + \hat{A}_\lambda &= \partial + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \lambda U_{0,N}^2 \\ U_{1,0}^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & U_{2,1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & U_{N,N-1}^2 & 0 \end{pmatrix} \\ \partial + \hat{B}_\lambda &= \partial + \begin{pmatrix} 2\bar{\partial}u_0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 2\bar{\partial}u_1 & 1 & \dots & 0 & 0 \\ 0 & 0 & 2\bar{\partial}u_2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2\bar{\partial}u_{N-1} & 1 \\ \lambda^{-1} & 0 & 0 & \dots & 0 & 2\bar{\partial}u_N \end{pmatrix} \end{aligned} \quad (3.9)$$

Let f_n be functions on a Riemann surface Σ (3.4) defined by (3.3). The Riemann surface forms a $N + 1$ sheet covering of the λ -plane. Away from the branch points, each point $P = \lambda$ on the λ -plane corresponds to $N + 1$ points $P_i = (y_i, \lambda)$ on the Riemann surface Σ . Since the f_n provides a solution to (3.7), if we let F_1 be the following matrix

$$F_1 = \begin{pmatrix} f_0(P_1) & f_0(P_2) & \dots & f_0(P_{N+1}) \\ f_1(P_1) & f_1(P_2) & \dots & f_1(P_{N+1}) \\ f_2(P_1) & f_2(P_2) & \dots & f_2(P_{N+1}) \\ \vdots & \vdots & \vdots & \vdots \\ f_{N-1}(P_1) & f_{N-1}(P_2) & \dots & f_{N-1}(P_{N+1}) \\ f_N(P_1) & f_N(P_2) & \dots & f_N(P_{N+1}) \end{pmatrix} \quad (3.10)$$

then the following equations will be satisfied

$$(\partial + \hat{A}_\lambda)F_1 = 0, \quad (\partial + \hat{B}_\lambda)F_1 = 0 \quad (3.11)$$

Most of the equations in the above follows from (3.7), The only non trivial ones are the ones that involve λ , that is

$$\begin{aligned} \partial f_0 &= -\lambda U_{0,N}^2 f_N \\ \bar{\partial} f_N &= \lambda^{-1} f_0 + \partial u_N f_N \end{aligned}$$

To see that this is true, we observe that λ^{-1} is a meromorphic function on Σ that has an order N pole at the point Q_0 and an order N zero at the point Q_∞ with no poles or zeros elsewhere. This means that $\lambda^{-1}f_0 = f_{N+1}$ and hence the above equations are true.

Since the hatted (\hat{A}_λ) connection (3.11) and the unhatted one is related by the gauge transformation \mathbb{G} , a fundamental solution F to the equations (2.4) can now be formed by using the f_n

$$F(\lambda) = \mathbb{G}^{-1}F_1$$

It remains to express $h_{n,0}^{(1)}$ in terms of theta functions. Since $\Theta(U(P) - U(Q_\infty) - e)$ and $\Theta(U(P) - U(Q_0) - e)$ have zero at Q_∞ and Q_0 respectively, we have

$$\begin{aligned}\Theta(U(P) - U(Q_\infty) - e) &= s_\infty k_\infty^{-1} + \dots, & P \rightarrow Q_\infty \\ \Theta(U(P) - U(Q_0) - e) &= s_0 k_0^{-1} + \dots, & P \rightarrow Q_0\end{aligned}$$

So the normalization constant in (3.3) is

$$\begin{aligned}C &= \left(\frac{\Theta(U(Q_0) - U(Q_\infty) - e)^n}{\Theta(U(Q_0) - U(D) - K)s_0^n} \right)^{-1} \\ &\times \Theta(U(Q_0) - U(D) + n(U(Q_0) - U(Q_\infty)) + \xi\mathbb{B}_1 + \bar{\xi}\mathbb{B}_2 - K)\end{aligned}$$

therefore $h_{n,0}^{(1)}$ is

$$h_{n,0}^{(1)} = \frac{\Theta(U(Q_\infty) - U(D) + n(U(Q_0) - U(Q_\infty)) + \xi\mathbb{B}_1 + \bar{\xi}\mathbb{B}_2 - K)}{\Theta(U(Q_0) - U(D) + n(U(Q_0) - U(Q_\infty)) + \xi\mathbb{B}_1 + \bar{\xi}\mathbb{B}_2 - K)} e^c e_n^c n$$

where c and c_n are constants

$$\begin{aligned}c &= \ln \frac{\Theta(U(Q_0) - U(D) - K)}{\Theta(U(Q_\infty) - U(D) - K)} \\ c_n &= \ln \frac{\Theta(U(Q_\infty) - U(Q_0) - e)s_0}{\Theta(U(Q_0) - U(Q_\infty) - e)s_\infty}\end{aligned}$$

Since $e^{2u_i} = |h_{n,0}^{(1)}|$, the proposition follows. \square

4 Relation to immersed surface

Let us now discuss the analytical description of a 2-dimensional surface immersed in the $\mathfrak{su}(N)$ algebra, associated with the generalized Weierstrass formula for the immersion given below by the expressions (4.4) and (4.7).

Suppose now one of the times, say t_m in the hierarchy is trivial, that is,

$$\partial_{t_m} A_\lambda = \partial_{t_m} B_\lambda = 0$$

then for the corresponding matrix L_m , we have the following compatibility conditions

$$\begin{aligned}\partial L_m &= [L_m, A_\lambda] \\ \bar{\partial} L_m &= [L_m, B_\lambda]\end{aligned}\tag{4.1}$$

By applying the involution ρ to equations (4.1), we get

$$\begin{aligned}\partial [\rho(L_m)] &= [\rho(L_m), A_\lambda] \\ \bar{\partial} [\rho(L_m)] &= [\rho(L_m), B_\lambda]\end{aligned}\tag{4.2}$$

Therefore the matrix

$$X = i(L_m + \rho(L_m)) \in \mathfrak{su}(N+1)$$

also satisfies the above equalities (4.2).

The spectral curve Σ is the zero set of the determinant

$$\det(y - (L_m(\lambda) + \rho(L_m(\lambda))))$$

It is a Riemann surface defined by a polynomial in λ and y of the following form

$$y^{N+1} = a_m \lambda^m + a_{n-1} \lambda^{n-1} + \bar{a}_m \lambda^{-m} + P(y, \lambda)$$

where $P(y, \lambda)$ is a polynomial in λ and y . We see that this curve has the form required by (3.4). Therefore we can apply the results of the previous section and express the solution of the Toda lattice in terms of theta function on the spectral curve.

If we set $|\lambda| = 1$, we have the following compatibility conditions

$$\partial(L_m + L_m^\dagger) = [L_m + L_m^\dagger, A_\lambda], \quad \bar{\partial}(L_m + L_m^\dagger) = [L_m + L_m^\dagger, B_\lambda]\tag{4.3}$$

We can define the $N \times N$ matrix

$$K(\lambda) = [L_m + L_m^\dagger, B_\lambda]\tag{4.4}$$

and its hermitian conjugate

$$K^\dagger(\lambda) = [L_m + L_m^\dagger, A_\lambda]$$

From (4.3) we obtain the following

$$\partial K - \bar{\partial} K^\dagger = 0\tag{4.5}$$

Hence $\partial K \in i\mathfrak{su}(N)$ is a hermitian matrix.

As a consequence of the conservation law (4.5), we will show in the next section that there exists real-valued functions $X_i(\xi, \bar{\xi})$ which can be identified with the Weierstrass representation of surfaces immersed in the multi-dimensional space \mathbb{R}^{N^2-1} .

By using the Drinfeld-Sokolov iteration process [12] introduced in section 2, the matrix

$$X = i(L_m + L_m^\dagger)$$

and hence the immersion, can be computed.

4.1 The generalized Weierstrass formula for immersion

In order to study immersions defined by means of theta function solutions of the Toda Lattice and to derive from this model the moving frames and the corresponding Gauss-Weingarten and the Gauss-Codazzi-Ricci equations, it is convenient to exploit the Euclidean structure of the $\mathfrak{su}(N)$ Lie algebra leading to an identification

$$\mathbb{R}^{N^2-1} \cong \mathfrak{su}(N)$$

Consequently, we can introduce on $\mathfrak{su}(N)$ an inner product $(,) : \mathfrak{su}(N) \times \mathfrak{su}(N) \rightarrow \mathbb{R}$ of vectors in terms of matrices

$$(X, Y) = -\frac{1}{2} \text{tr}(XY), \quad X, Y \in \mathfrak{su}(N)$$

Let us assume that the matrix K is constructed from a solution of the equation of motion written in the form of the conservation law (4.5). This conservation law implies that the matrix-valued 1-form

$$dX = i(K^\dagger d\xi + K d\bar{\xi}) \quad (4.6)$$

is closed ($d(dX) = 0$) and it takes values in the Lie algebra $\mathfrak{su}(N)$ of the anti-hermitian matrices. The real and the imaginary parts of dX are anti-symmetric and symmetric, respectively

$$\begin{aligned} dX &= dX^1 + i dX^2 \\ (dX^1)^T &= -dX^1, \quad (dX^2)^T = dX^2 \end{aligned}$$

where the 1-forms dX^1 and dX^2 take values in $\mathfrak{sl}(N, \mathbb{R})$.

From the closeness of dX it follows that the integral

$$X(\xi, \bar{\xi}) = i \int_{\gamma} (K^\dagger d\xi + K d\bar{\xi}) \quad (4.7)$$

locally depends only on the end points of the curve γ in \mathbb{C} .

The integral defines the mapping

$$X : \Omega \in \mathbb{C} \rightarrow X(\xi, \bar{\xi}) \in \mathfrak{su}(N) \cong \mathbb{R}^{N^2-1}$$

which is called the generalized Weierstrass formula for immersion (GWFI). The complex tangent vectors of this immersion, by virtue of (4.6), are

$$\partial X = iK^\dagger, \quad \bar{\partial} X = iK$$

Hence a surface \mathcal{F} associated with the Toda model by means of the immersion (4.4) satisfies the following relations

$$\begin{aligned} \partial X &= i[(L_m^\dagger + L_m), B_\lambda] \\ \bar{\partial} X &= i[(L_m^\dagger + L_m), A_\lambda] \end{aligned}$$

The components of the metric on \mathcal{F} induced by the Euclidean structure in $\mathfrak{su}(N)$ are given by

$$\begin{aligned} g_{11} &= (\partial X, \partial X) = \frac{1}{2} \text{tr}(\partial(L_m + L_m^\dagger) \partial(L_m + L_m^\dagger)) \\ g_{12} &= g_{21} = (\partial X, \bar{\partial} X) = -\frac{1}{2} \text{tr}(\partial(L_m + L_m^\dagger) \bar{\partial}(L_m + L_m^\dagger)) \\ g_{22} &= (\bar{\partial} X, \bar{\partial} X) = \frac{1}{2} \text{tr}(\bar{\partial}(L_m + L_m^\dagger) \bar{\partial}(L_m + L_m^\dagger)) \end{aligned}$$

The first fundamental form of the surface \mathcal{F} takes the form

$$I = g_{11}(\mathrm{d}\xi)^2 + 2g_{21}\mathrm{d}\xi\mathrm{d}\bar{\xi} + g_{22}(\mathrm{d}\bar{\xi})^2 \quad (4.8)$$

and the second fundamental form is

$$II = (\partial^2 X)^\perp \mathrm{d}\xi \mathrm{d}\xi + 2(\partial \bar{\partial} X)^\perp \mathrm{d}\xi \mathrm{d}\bar{\xi} + (\bar{\partial}^2 X)^\perp \mathrm{d}\bar{\xi} \mathrm{d}\bar{\xi} \quad (4.9)$$

where A^\perp is the normal component of A to the surface \mathcal{F} . The mean curvature vector is given by

$$H = \det G^{-1} (g_{22}(\partial^2 X)^\perp - 2g_{12}(\partial \bar{\partial} X)^\perp + g_{11}(\bar{\partial}^2 X)^\perp)$$

where G is the matrix formed by the metric $G_{ij} = g_{ij}$.

The Gaussian curvature is not necessarily constant

$$\begin{aligned} K &= (2 \det G)^{-1} \left(\partial \left(\frac{1}{\det G} (g_{12} \bar{\partial} \ln g_{11} - \partial g_{22}) \right) \right. \\ &\quad \left. + \bar{\partial} \left(\frac{1}{\det G} (2\partial g_{12} - \bar{\partial} g_{11} - g_{12} \partial \ln g_{11}) \right) \right) \end{aligned}$$

It is easy to check that

$$\begin{aligned} \text{tr}(\partial \bar{\partial} L \partial L) &= \text{tr}([L, \bar{\partial} A][L, A]) \\ \text{tr}(\partial \bar{\partial} L \bar{\partial} L) &= \text{tr}([L, \partial B][L, B]) = \text{tr}([L, \bar{\partial} A][L, B]) \end{aligned}$$

By equation (4.1), we see that

$$\partial X = [X, A_\lambda], \quad \bar{\partial} X = [X, B_\lambda]$$

Hence we have

$$k \partial \text{tr}(X^k) = \text{tr}(X^{k-1} [X, A_\lambda]), \quad = \text{tr}([X^k, A_\lambda]) = 0 \quad (4.10)$$

therefore the invariants of the surface is generated by $\text{tr}(X^i)$, $i = 2, \dots, m$. In particular, these surfaces are all mapped into hyperspheres immersed in \mathbb{R}^{N^2-1} .

5 The immersion in the $\mathfrak{su}(N)$ case

In the $\mathfrak{su}(N)$ case, the matrix $L_n(\lambda)$ can be computed as follows. First we make the gauge transformation

$$\begin{aligned}\hat{A}_\lambda &= \mathbb{G}^{-1}A_\lambda\mathbb{G} + \mathbb{G}^{-1}\partial\mathbb{G} = \begin{pmatrix} 2\partial u_0 & 0 & 0 & \dots & 0 & \lambda \\ 1 & 2\partial u_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2\partial u_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2\partial u_{N-1} & 0 \\ 0 & 0 & 0 & \dots & 1 & 2\partial u_N \end{pmatrix} \\ \mathbb{G} &= \text{diag}(e^{u_0}, e^{u_1}, \dots, e^{u_N})\end{aligned}$$

We can then expand \hat{A}_λ and $L_n(\lambda)$ in terms of Λ

$$\begin{aligned}\hat{A}_\lambda &= \begin{pmatrix} 2\partial u_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2\partial u_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2\partial u_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2\partial u_{N-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 2\partial u_N \end{pmatrix} + \Lambda \\ L_n(\lambda) &= \sum_{i=0}^{3n+1} l_{3n+1-i} \Lambda^i\end{aligned}$$

where l_i are diagonal matrices.

The compatibility condition (4.1) becomes

$$\begin{aligned}\partial L_n(\lambda) &= [L_n(\lambda), \Lambda + \hat{A}_0] \\ \hat{A}_0 &= 2\text{diag}(\partial u_0, \dots, \partial u_N)\end{aligned}\tag{5.1}$$

The expansion of (5.1) in terms of Λ then takes the following form

$$\sum_{i=1}^{3n+1} \partial l_{3n+1-i} \Lambda^i = \sum_{i=1}^{3n+1} (l_{3n+1-i} \Lambda^{i+1} - \Lambda l_{3n+1-i} \Lambda^i - \hat{A}_0 l_{3n+1-i} \Lambda^i + l_{3n+1-i} \Lambda^i \hat{A}_0)$$

Let σ be the permutation such that

$$(\sigma(0), \sigma(1), \dots, \sigma(N)) = (N, 0, 1, \dots, N-1)$$

then we have

$$\Lambda \begin{pmatrix} a_0 & \dots & \dots & 0 \\ 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_N \end{pmatrix} = \begin{pmatrix} a_{\sigma(0)} & \dots & \dots & 0 \\ 0 & a_{\sigma(1)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_{\sigma(N)} \end{pmatrix} \Lambda$$

By using this permutation, we can express (5.1) in terms of the coefficients l_i .

$$l_{i+1} - \sigma(l_{i+1}) = \partial l_i - l_i(\sigma^{3n+1-i}(A_0) - A_0) \quad (5.2)$$

where the action of σ on a diagonal matrix is defined as follows

$$\sigma \text{diag}(m_0, m_2, \dots, m_N) = \text{diag}(m_{\sigma(0)}, m_{\sigma(1)}, \dots, m_{\sigma(N)})$$

The equation (5.2) determines the coefficients l_i uniquely up to an addition of a scalar provided the right hand side is traceless. The condition that the right hand side of (5.2) is traceless then fixes the scalar freedom in l_i .

For example, in the case of $\mathfrak{su}(3)$ and $n = 1$, we have the explicit form of the coefficients l_i

$$\begin{aligned} l_0 &= I_d \\ l_1 &= \begin{pmatrix} 2\partial u_0 & 0 & 0 \\ 0 & 2\partial u_1 & 0 \\ 0 & 0 & 2\partial u_2 \end{pmatrix} \\ l_2 &= \begin{pmatrix} \frac{2}{3}(\partial^2 u_0 - \partial^2 u_1 - c_2) & 0 & 0 \\ 0 & \frac{2}{3}(\partial^2 u_1 - 2\partial^2 u_2 - c_2) & 0 \\ 0 & 0 & \frac{2}{3}(2\partial^2 u_2 - \partial^2 u_0 - c_2) \end{pmatrix} \\ c_i &= (\partial u_0)^i + (\partial u_1)^i + (\partial u_2)^i \\ l_3 &= \text{diag}(l_3^1, l_3^2, l_3^3) \\ l_3^1 &= \frac{1}{3} \left(-2\partial^3 u_1 - \partial c_2 - 4\partial^2 u_0(\partial u_1 - \partial u_0) + 4\partial u_1 c_2 - \frac{4}{3}c_3 \right) \\ l_3^2 &= \frac{1}{3} \left(-2\partial^3 u_2 - 3\partial c_2 - 4\partial^2 u_0(\partial u_1 - \partial u_0) - \frac{4}{3}c_3 + 2\partial(\partial u_2 - \partial u_1)^2 + 4\partial u_2 c_2 \right) \\ l_3^3 &= \frac{1}{3} \left(-2\partial^3 u_0 + \partial c_2 - 4\partial^2 u_0(\partial u_1 - \partial u_0) - \frac{4}{3}c_3 - 2\partial(\partial u_0 - \partial u_1)^2 - 4\partial u_0 c_2 \right) \\ l_4 &= \text{diag}(l_4^1, l_4^2, l_4^3) \\ l_4^1 &= -\frac{1}{3} ((\partial l_3^1 + 2(\partial u_0 - \partial u_2)l_3^1) - (\partial l_3^2 + 2(\partial u_1 - \partial u_0)l_3^2)) \\ l_4^2 &= \frac{1}{3} (-(\partial l_3^1 + 2(\partial u_0 - \partial u_2)l_3^1) + 2(\partial l_3^2 + 2(\partial u_1 - \partial u_0)l_3^2)) \\ l_4^3 &= \frac{1}{3} (-2(\partial l_3^1 + 2(\partial u_0 - \partial u_2)l_3^1) + (\partial l_3^2 + 2(\partial u_1 - \partial u_0)l_3^2)) \end{aligned} \quad (5.3)$$

5.1 Moving frame in the $\mathfrak{su}(N)$ case

In this section we will construct a moving frame for the immersed surface \mathcal{F} in the $\mathfrak{su}(N)$ case, which will then be used to compute the Gauss-Weingarten equations.

To begin with, let the action of the involution ρ (3.5) on a matrix-valued function $\Phi(P)$ on the spectral curve be defined as follows

$$\rho(\Phi(P)) = \Phi^\dagger(\rho(P)) \quad (5.4)$$

We then have the following

Lemma 5.1 *Let F_1 be the matrix in (3.10), and $\Phi = \mathbb{G}F_1$, where*

$$\mathbb{G} = \text{diag}(e^{u_0}, e^{u_1}, \dots, e^{u_N})$$

then

$$\begin{aligned} \rho(\Phi)\Phi &= \text{diag}(d_1, \dots, d_{N+1}) \\ &= \mathbb{D} \\ d_j &= \sum_{i=0}^N e^{u_i + \overline{u_i}} f_i(P_j) \overline{f_i}(\rho(P_j)) \end{aligned} \quad (5.5)$$

Proof. Let $\Phi = \mathbb{G}F_1$, $H_1 = \rho(\Phi)$ and $H_2 = (\Phi)^{-1}$. First note that both H_1 and H_2 are matrices that has the following asymptotic behavior

$$\begin{aligned} H_j &= \exp D_0 T_0^j \lambda^S, \quad \lambda \rightarrow 0 \\ H_j &= \exp D_\infty T_\infty^j \lambda^S, \quad \lambda \rightarrow \infty \\ D_0 &= -\xi \lambda^{\frac{1}{N+1}} \text{diag}(1, \omega, \omega^2, \dots, \omega^N) \\ D_\infty &= \overline{\xi} \lambda^{-\frac{1}{N+1}} \text{diag}(1, \omega, \omega^2, \dots, \omega^N) \\ S &= \text{diag}\left(0, \frac{1}{N+1}, \dots, \frac{N}{N+1}\right) \\ \omega &= e^{\frac{2\pi i}{N+1}} \end{aligned} \quad (5.6)$$

where T_0^i and T_∞^i are power series in $\lambda^{\frac{1}{N+1}}$ and $\lambda^{-\frac{1}{N+1}}$ that are invertible at 0 and ∞ respectively.

By applying the involution ρ to the equations

$$\partial\Phi + A_\lambda\Phi = 0, \quad \overline{\partial}\Phi + B_\lambda\Phi = 0$$

we obtain

$$\overline{\partial}H_1 - H_1 B_\lambda = 0, \quad \partial H_1 - H_1 A_\lambda = 0 \quad (5.7)$$

Similarly, by differentiating Φ^{-1} , we see that H_2 satisfies the same equations

$$\overline{\partial}H_2 - H_2 B_\lambda = 0, \quad \partial H_2 - H_2 A_\lambda = 0$$

Since any solution to the equations (5.7) with the asymptotic behavior (5.6) are determined uniquely up to the multiplication of a diagonal matrix \mathbb{D} on the left, we have

$$H_1 = \mathbb{D}H_2$$

for some diagonal matrix \mathbb{D} constant in ξ and $\overline{\xi}$. Therefore we have

$$H_1\Phi = \rho(F_1)\overline{\mathbb{G}}\mathbb{G}F_1 = \mathbb{D}$$

By computing the diagonal entries in $\rho(F_1)\overline{\mathbb{G}}\mathbb{G}F_1$, the lemma is proven. \square

We can now construct the moving frame for the surface \mathcal{F} immersed in $\mathfrak{su}(N+1)$. First note that the function $T(P) = \Phi(P) \exp(-D_0 - D_\infty)$, where Φ , D_0 and D_∞ are defined in (5.6), satisfies the following

$$\partial TT^{-1} + A_\lambda = TD_A T^{-1}, \quad \bar{\partial} TT^{-1} + B_\lambda = TD_B T^{-1}$$

where D_A and D_B are diagonal matrices. We see that T and the dressing matrix T_∞ in (2.5) are related by

$$T = T_\infty \Psi$$

where Ψ is the matrix that diagonalizes Λ .

Since L_m in (4.1) are defined by (2.7)

$$L_i = (T_\infty \Lambda^i T_\infty^{-1})_+ = (TDT^{-1}) + \partial_m TT^{-1} = (TDT^{-1})$$

for some diagonal matrix D that is constant in ξ and $\bar{\xi}$. The second equality follows as $\partial_m T = 0$. Therefore T , and hence Φ , diagonalizes L_m , and hence X .

We can now take the conjugation of A_λ , B_λ and X by the matrix Φ to obtain the following

$$\begin{aligned} X &= \Phi Y \Phi^{-1} \\ Y &= \text{diag}(y_1, \dots, y_{N+1}) \\ A_\lambda &= \Phi \Xi_1 \Phi^{-1}, \quad B_\lambda = \Phi \Xi_2 \Phi^{-1} \\ (\Xi_1)_{kl} &= - \left(\sum_{i=0}^N \phi_i(P_k) \bar{\phi}_i(\rho(P_k)) \right)^{-1} \sum_{i=0}^N \bar{\phi}_i(\rho(P_k)) \partial \phi_i(P_l) \\ (\Xi_2)_{kl} &= - \left(\sum_{i=0}^N \phi_i(P_k) \bar{\phi}_i(\rho(P_k)) \right)^{-1} \sum_{i=0}^N \bar{\phi}_i(\rho(P_k)) \bar{\partial} \phi_i(P_l) \\ \phi_i(P) &= e_i^u f_i(P) \end{aligned} \quad (5.8)$$

the entries y_i of Y are the different branches of the spectrum y in the spectral curve (3.4). Note that they do not depend on ξ and $\bar{\xi}$.

The tangent vectors ∂X and $\bar{\partial} X$ can be represented as

$$\begin{aligned} \partial X &= [X, A_\lambda] = \Phi \partial X^0 \Phi^{-1}, \quad \bar{\partial} X = [X, B_\lambda] = \Phi \bar{\partial} X^0 \Phi^{-1} \\ (\partial X^0)_{kl} &= (y_k - y_l) (\Xi_1)_{kl} \\ (\bar{\partial} X^0)_{kl} &= (y_k - y_l) (\Xi_2)_{kl} \end{aligned} \quad (5.9)$$

From this the metric and the first fundamental form are

$$g_{11} = -\frac{1}{2} \text{tr}(\partial X \partial X) = -\frac{1}{2} \sum_{k,l=1}^{N+1} (y_k - y_l)^2 (\Xi_1)_{kl} (\Xi_1)_{lk}$$

$$\begin{aligned}
g_{12} &= g_{21} = -\frac{1}{2}\text{tr}(\partial X \bar{\partial} X) = -\frac{1}{2} \sum_{k,l=1}^{N+1} (y_k - y_l)^2 (\Xi_1)_{kl} (\Xi_2)_{lk} \quad (5.10) \\
g_{22} &= -\frac{1}{2}\text{tr}(\bar{\partial} X \partial X) = -\frac{1}{2} \sum_{k,l=1}^{N+1} (y_k - y_l)^2 (\Xi_2)_{kl} (\Xi_1)_{lk} \\
I &= g_{11} d\xi d\xi + 2g_{12} d\xi d\bar{\xi} + g_{22} d\bar{\xi} d\bar{\xi}
\end{aligned}$$

Let E_{ij} be the $(N+1) \times (N+1)$ matrix that is 1 in its ij^{th} entry and zero elsewhere. The following forms a basis of the $\mathfrak{su}(N+1)$ algebra

$$\begin{aligned}
s(k, l) &= \frac{(k-2)(k-1)}{2} + l \\
v_{s(k,l)}^0 &= E_{kl} - E_{lk}, \quad l < k, \quad k = 2, \dots, N+1 \\
u_{s(k,l)}^0 &= i(E_{kl} + E_{lk}), \quad l < k, \quad k = 2, \dots, N+1 \\
d_k^0 &= i(E_{11} - E_{kk}), \quad k = 2, \dots, N+1 \\
v_{s(k,l)} &= \Phi v_{s(k,l)}^0 \Phi^{-1} \\
u_{s(k,l)} &= \Phi u_{s(k,l)}^0 \Phi^{-1} \\
d_k &= \Phi d_k^0 \Phi^{-1}
\end{aligned}$$

We could now construct an orthonormal basis to the surface in terms of these basis vectors.

Theorem 5.2 *The following forms an orthogonal basis of the normal vectors to the surface \mathcal{F} .*

$$\begin{aligned}
U_l &= \sum_{i=1}^l C_i^l u_i, \quad l = 3, \dots, \frac{N(N+1)}{2} \\
V_l &= \sum_{i=1}^l D_i^l v_i, \quad l = 3, \dots, \frac{N(N+1)}{2} \\
W_0 &= X \\
W_l &= \sum_{i=2}^l K_i^l d_i
\end{aligned} \quad (5.11)$$

where the constants C_i^l , D_i^l and K_i^l are given by the following

$$\begin{aligned}
C_i^l &= 0, i > l \\
C_i^l &= \det \begin{pmatrix} J_1 & J_2 & \dots & J_{i-1} & -J_l & J_{i+1} & \dots & J_{l-1} \\ -\bar{J}_1 & -\bar{J}_2 & \dots & -\bar{J}_{i-1} & \bar{J}_l & -\bar{J}_{i+1} & \dots & -\bar{J}_{l-1} \\ C_1^{l-1} & C_2^{l-1} & \dots & C_{i-1}^{l-1} & 0 & C_{i+1}^{l-1} & \dots & C_{l-1}^{l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ C_1^3 & C_2^3 & \dots & C_{i-1}^3 & 0 & C_{i+1}^3 & \dots & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
C_l^l &= \det \begin{pmatrix} J_1 & J_2 & \dots & J_{l-1} \\ -\bar{J}_1 & -\bar{J}_2 & \dots & -\bar{J}_{l-1} \\ C_1^{l-1} & C_2^{l-1} & \dots & C_{l-1}^{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ C_1^3 & C_2^3 & \dots & 0 \end{pmatrix} \\
D_i^l &= \det \begin{pmatrix} Q_1 & Q_2 & \dots & Q_{i-1} & -Q_l & Q_{i+1} & \dots & Q_{l-1} \\ -\bar{Q}_1 & -\bar{Q}_2 & \dots & -\bar{Q}_{i-1} & \bar{Q}_l & -\bar{Q}_{i+1} & \dots & -\bar{Q}_{l-1} \\ D_1^{l-1} & D_2^{l-1} & \dots & D_{i-1}^{l-1} & 0 & D_{i+1}^{l-1} & \dots & D_{l-1}^{l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ D_1^3 & D_2^3 & \dots & D_{i-1}^3 & 0 & D_{i+1}^3 & \dots & 0 \end{pmatrix} \\
D_l^l &= \det \begin{pmatrix} Q_1 & Q_2 & \dots & Q_{l-1} \\ -\bar{Q}_1 & -\bar{Q}_2 & \dots & -\bar{Q}_{l-1} \\ D_1^{l-1} & D_2^{l-1} & \dots & D_{l-1}^{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ D_1^3 & D_2^3 & \dots & 0 \end{pmatrix} \\
J_{s(k,l)} &= -\frac{1}{2} \text{tr}(u_{s(k,l)} \partial X) = -\frac{i}{2} ((\partial X^0)_{lk} + (\partial X^0)_{kl}) \\
Q_{s(k,l)} &= -\frac{1}{2} \text{tr}(v_{s(k,l)} \partial X) = -\frac{1}{2} ((\partial X^0)_{lk} - (\partial X^0)_{kl}) \\
K_k^l &= \det \begin{pmatrix} R_1 & R_2 & \dots & R_{k-1} & -R_l & R_{k+1} & \dots & R_{l-1} \\ Z_1 & Z_2 & \dots & Z_{k-1} & -1 & Z_{k+1} & \dots & Z_{l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -2K_2^2 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \end{pmatrix} \\
Z_1 &= -2K_2^{l-1} + \sum_{j=3}^{l-1} K_j^{l-1} \\
Z_k &= -2K_{k-1}^{l-1} + K_2^{l-1} \\
R_k &= \text{tr}(d_k X) = i(y_1 - y_k)
\end{aligned}$$

The coefficients in (5.11) can be computed by solving systems of linear equations. To compute the first coefficient C_1^l , we need to solve

$$\begin{aligned}
a_1 \text{tr}(\partial X v_1) + a_2 \text{tr}(\partial X v_2) + \text{tr}(\partial X v_3) &= 0 \\
a_1 \text{tr}(\bar{\partial} X v_1) + a_2 \text{tr}(\bar{\partial} X v_2) + \text{tr}(\bar{\partial} X v_3) &= 0
\end{aligned}$$

for a_1 and a_2 . The solution is given by

$$a_1 = -\frac{\det \begin{pmatrix} J_3 & J_2 \\ \bar{J}_3 & \bar{J}_2 \end{pmatrix}}{\det \begin{pmatrix} J_1 & J_2 \\ \bar{J}_1 & \bar{J}_2 \end{pmatrix}}$$

$$a_2 = -\frac{\det\left(\frac{J_1}{J_1} \quad \frac{J_3}{J_3}\right)}{\det\left(\frac{J_1}{J_1} \quad \frac{J_2}{J_2}\right)}$$

Then, by multiplying the vector

$$a_1 v_1 + a_2 v_2 + v_3$$

by the common dominator of a_1 and a_2 , one sees that

$$V_1 = -\det\left(\frac{J_3}{J_3} \quad \frac{J_2}{J_2}\right) v_1 - \det\left(\frac{J_1}{J_1} \quad \frac{J_3}{J_3}\right) v_2 + \det\left(\frac{J_1}{J_1} \quad \frac{J_2}{J_2}\right) v_3$$

is a normal vector to the surface \mathcal{F} . To compute the other coefficients, one needs to solve linear systems of equations

$$\begin{aligned} a_1 \operatorname{tr}(\partial X v_1) + \dots + a_{l+1} \operatorname{tr}(\partial X v_{l-1}) + \operatorname{tr}(\partial X v_l) &= 0 \\ a_1 \operatorname{tr}(\bar{\partial} X v_1) + \dots + a_{l+1} \operatorname{tr}(\bar{\partial} X v_{l-1}) + \operatorname{tr}(\bar{\partial} X v_l) &= 0 \\ a_1 \operatorname{tr}(V_1 v_1) + \dots + a_{l+1} \operatorname{tr}(V_1 v_{l-1}) + \operatorname{tr}(V_1 v_l) &= 0 \\ &\dots \\ a_1 \operatorname{tr}(V_{l-1} v_1) + \dots + a_{l+1} \operatorname{tr}(V_{l-1} v_{l-1}) + \operatorname{tr}(V_{l-1} v_l) &= 0 \end{aligned}$$

and similar equations for the vectors U_i and W_i .

5.2 The Weingarten equation in a non-orthonormal basis

We can construct another basis of the normal vectors as follows. Let a_{i1} and a_{i2} be the solution of the following linear equations

$$\begin{aligned} a_{i1} \operatorname{tr}(u_1 \partial X) + a_{i2} \operatorname{tr}(v_1 \partial X) + \operatorname{tr}(\partial X v_i) &= 0 \\ a_{i1} \operatorname{tr}(u_1 \bar{\partial} X) + a_{i2} \operatorname{tr}(v_1 \bar{\partial} X) + \operatorname{tr}(\bar{\partial} X v_i) &= 0 \end{aligned}$$

Then a_{i1} and a_{i2} are given by the following

$$\begin{aligned} a_{i1} &= -\frac{\det\left(\frac{Q_i}{Q_i} \quad \frac{Q_1}{Q_1}\right)}{\det\left(\frac{J_1}{J_1} \quad \frac{Q_1}{Q_1}\right)} \\ a_{i2} &= -\frac{\det\left(\frac{J_1}{J_1} \quad \frac{Q_i}{Q_i}\right)}{\det\left(\frac{J_1}{J_1} \quad \frac{Q_1}{Q_1}\right)} \end{aligned}$$

By similar argument as before, we see that the vectors

$$\begin{aligned}
n_j^v &= (\gamma_{1,1}v_{j+1} - \gamma_{1,j+1}v_1 - \kappa_{j+1,1}u_1), \quad j = 1, \dots, \frac{N(N-1)}{2} \\
n_j^u &= (\gamma_{1,1}u_{j+1} - \beta_{1,j+1}v_1 - \gamma_{j+1,1}u_1), \quad j = 1, \dots, \frac{N(N-1)}{2} \\
\beta_{k,l} &= i \det \begin{pmatrix} J_k & J_l \\ \bar{J}_k & \bar{J}_l \end{pmatrix} \\
\gamma_{k,l} &= i \det \begin{pmatrix} J_k & Q_l \\ \bar{J}_k & \bar{Q}_l \end{pmatrix} \\
\kappa_{k,l} &= i \det \begin{pmatrix} Q_k & Q_l \\ \bar{Q}_k & \bar{Q}_l \end{pmatrix} \\
n_j^d &= i\Phi(E_{11} - E_{j+1,j+1})\Phi^{-1}, \quad j = 1, \dots, N-1 \\
n_N^d &= iy_{N+1}^{-1}X
\end{aligned} \tag{5.12}$$

form a basis of normal vectors to the surface \mathcal{F} . Although this basis is not orthogonal, they are a lot simpler than the orthonormal basis (5.11) because the coefficients only involves determinants of 2×2 matrices. Note that n_j^v , n_j^u and n_j^d all belong to $\mathfrak{su}(N+1)$.

We will now calculate the Gauss-Weingarten equation with this basis of normal vectors. By differentiating the vector n_j^v with respect to ξ we see that

$$\begin{aligned}
\partial n_j^v &= [\partial\Phi\Phi^{-1}, n_j^v] + (\partial\gamma_{1,1}v_{j+1} - \partial\gamma_{1,j+1}v_1 - \partial\kappa_{j+1,1}u_1) \\
&= [n_j^v, A_\lambda] + (\partial\gamma_{1,1}v_{j+1} - \partial\gamma_{1,j+1}v_1 - \partial\kappa_{j+1,1}u_1)
\end{aligned}$$

since $\partial\Phi\Phi^{-1} = -A_\lambda$. By solving similar linear equations, we see that the tangent component of the ∂n_j^v is given by

$$\begin{aligned}
(\partial n_j^v)_T &= \nu_{j,1}\partial X + \nu_{j,2}\bar{\partial}X \\
\nu_{j1} &= \det G^{-1} \det \begin{pmatrix} b_{j1} & g_{11} \\ b_{j2} & g_{12} \end{pmatrix} \\
\nu_{j2} &= \det G^{-1} \det \begin{pmatrix} g_{12} & b_{j1} \\ \bar{g}_{11} & b_{j2} \end{pmatrix} \\
b_{j1} &= \partial\gamma_{1,1}Q_{j+1} - \partial\gamma_{1,j+1}Q_1 - \partial\kappa_{j+1,1}J_1 - \frac{1}{2}\text{tr}([n_j^v, A_\lambda]\partial X) \\
b_{j2} &= \partial\gamma_{1,1}\bar{Q}_{j+1} - \partial\gamma_{1,j+1}\bar{Q}_1 - \partial\kappa_{j+1,1}\bar{J}_1 - \frac{1}{2}\text{tr}([n_j^v, A_\lambda]\bar{\partial}X)
\end{aligned} \tag{5.13}$$

Let η be a normal vector to the surface \mathcal{F} , then η can be written in terms of the basis (5.12) as follows

$$\eta = \sum_{j=1}^{\frac{N(N-1)}{2}} \eta_j^v n_j^v + \sum_{j=1}^{\frac{N(N-1)}{2}} \eta_j^u n_j^u + \sum_{j=1}^N \eta_j^d n_j^d$$

$$\begin{aligned}
&= \sum_{j=1}^{\frac{N(N-1)}{2}} \gamma_{1,1} \eta_j^v v_{j+1} + \sum_{j=1}^{\frac{N(N-1)}{2}} \gamma_{1,1} \eta_j^u u_{j+1} + c_0^u u_1 + c_0^v v_1 \\
&+ \Phi \left(\sum_{j=1}^N i \eta_j^d E_{j+1,j+1} + c_1^d E_{11} \right) \Phi^{-1}
\end{aligned}$$

Therefore the coefficients in this decomposition are given by

$$\begin{aligned}
\eta_i^v &= -\frac{1}{2\gamma_{1,1}} \text{tr}(\eta v_{i+1}), \quad \eta_i^u = -\frac{1}{2\gamma_{1,1}} \text{tr}(\eta u_{i+1}), \\
\eta_j^d &= -i \text{tr}(\eta \Phi E_{j+1,j+1} \Phi^{-1})
\end{aligned} \tag{5.14}$$

By subtracting the tangent component (5.13) from the vector ∂n_j^v , we obtain the normal component of ∂n_j^v

$$(\partial n_j^v)_\perp = [n_j^v, A_\lambda] + \partial \gamma_{1,1} v_{j+1} - \partial \gamma_{1,j+1} v_1 - \partial \kappa_{j+1,1} u_1 - \nu_{j1} \partial X - \nu_{j2} \bar{\partial} X$$

By using (5.14), the coefficients of this vector are given by the following

$$\begin{aligned}
(\partial n_j^v)_k^v &= \gamma_{1,1}^{-1} \left(-\frac{1}{2} \text{tr}([n_j^v, A_\lambda] v_{k+1}) + \partial \gamma_{1,1} \delta_{kj} - \nu_{j1} Q_{k+1} - \nu_{j2} \bar{Q}_{k+1} \right) \\
(\partial n_j^v)_k^u &= \gamma_{1,1}^{-1} \left(-\frac{1}{2} \text{tr}([n_j^v, A_\lambda] u_{k+1}) - \nu_{j1} J_{k+1} - \nu_{j2} \bar{J}_{k+1} \right) \\
(\partial n_j^v)_k^d &= -i \text{tr}([n_j^v, A_\lambda] \Phi E_{k+1,k+1} \Phi^{-1})
\end{aligned} \tag{5.15}$$

We can perform similar computations with derivatives of other normal vectors to obtain the Gauss-Weingarten equation.

Theorem 5.3 *The Gauss-Weingarten equation in terms of the basis*

$$\eta = (\partial X, \bar{\partial} X, n_j^v, n_j^u, n_j^d)^T$$

defined in (5.12) is given by

$$\begin{aligned}
\partial^2 X &= \alpha_{1,1} \partial X + \alpha_{1,2} \bar{\partial} X + \sum_{k=1}^{\frac{N(N-1)}{2}} ((\partial^2 X)_k^v n_k^v + (\partial^2 X)_k^u n_k^u) \\
&+ \sum_{k=1}^N (\partial^2 X)_k^d n_k^d \\
\partial \bar{\partial} X &= \alpha_{2,1} \partial X + \alpha_{2,2} \bar{\partial} X + \sum_{k=1}^{\frac{N(N-1)}{2}} ((\partial \bar{\partial} X)_k^v n_k^v + (\partial \bar{\partial} X)_k^u n_k^u) \\
&+ \sum_{k=1}^N (\partial \bar{\partial} X)_k^d n_k^d
\end{aligned}$$

$$\begin{aligned}
\partial n_j^v &= \nu_{j,1} \partial X + \nu_{j,2} \bar{\partial} X + \sum_{k=1}^{\frac{N(N-1)}{2}} ((\partial n_j^v)_k^v n_k^v + (\partial n_j^v)_k^u n_k^u) \\
&+ \sum_{k=1}^N (\partial n_j^v)_k^d n_k^d, \quad j = 1, \dots, \frac{N(N-1)}{2} \\
\partial n_j^u &= \mu_{j,1} \partial X + \mu_{j,2} \bar{\partial} X + \sum_{k=1}^{\frac{N(N-1)}{2}} ((\partial n_j^u)_k^v n_k^v + (\partial n_j^u)_k^u n_k^u) \\
&+ \sum_{k=1}^N (\partial n_j^u)_k^d n_k^d, \quad j = 1, \dots, \frac{N(N-1)}{2} \\
\partial n_j^d &= \chi_{j,1} \partial X + \chi_{j,2} \bar{\partial} X + \sum_{k=1}^{\frac{N(N-1)}{2}} ((\partial n_j^d)_k^v n_k^v + (\partial n_j^d)_k^u n_k^u) \\
&+ \sum_{k=1}^N (\partial n_j^d)_k^d n_k^d, \quad j = 1, \dots, N-1 \\
\partial n_j^N &= i y_{N+1}^{-1} \partial X
\end{aligned}$$

where the coefficients are

$$\begin{aligned}
\alpha_{1,1} &= (\det G)^{-1} \left(\frac{1}{2} \partial g_{11} \bar{g}_{11} - \partial g_{12} g_{12} + \frac{1}{2} \bar{\partial} g_{11} g_{12} \right) \\
\alpha_{1,2} &= (\det G)^{-1} \left(g_{11} \partial g_{12} - \frac{1}{2} g_{11} \bar{\partial} g_{11} - \frac{1}{2} g_{12} \partial g_{11} \right) \\
\alpha_{2,1} &= (2 \det G)^{-1} (\bar{\partial} g_{11} \bar{g}_{11} - \partial \bar{g}_{11} g_{12}) \\
\alpha_{2,2} &= (2 \det G)^{-1} (g_{11} \partial \bar{g}_{11} - \bar{\partial} g_{11} g_{12}) \\
\nu_{j1} &= \det G^{-1} \det \begin{pmatrix} b_{j1} & g_{11} \\ b_{j2} & g_{12} \end{pmatrix} \\
\nu_{j2} &= \det G^{-1} \det \begin{pmatrix} g_{12} & b_{j1} \\ \bar{g}_{11} & b_{j2} \end{pmatrix} \\
b_{j1} &= \partial \gamma_{1,1} Q_{j+1} - \partial \gamma_{1,j+1} Q_1 - \partial \kappa_{j+1,1} J_1 - \frac{1}{2} \text{tr}([n_j^v, A_\lambda] \partial X) \\
b_{j2} &= \partial \gamma_{1,1} \bar{Q}_{j+1} - \partial \gamma_{1,j+1} \bar{Q}_1 - \partial \kappa_{j+1,1} \bar{J}_1 - \frac{1}{2} \text{tr}([n_j^v, A_\lambda] \bar{\partial} X) \\
\mu_{j1} &= \det G^{-1} \det \begin{pmatrix} a_{j1} & g_{11} \\ a_{j2} & g_{12} \end{pmatrix} \\
\mu_{j2} &= \det G^{-1} \det \begin{pmatrix} g_{12} & a_{j1} \\ \bar{g}_{11} & a_{j2} \end{pmatrix} \\
a_{j1} &= \partial \gamma_{1,1} J_{j+1} - \partial \beta_{1,j+1} Q_1 - \partial \gamma_{j+1,1} J_1 - \frac{1}{2} \text{tr}([n_j^u, A_\lambda] \partial X) \\
a_{j2} &= \partial \gamma_{1,1} \bar{J}_{j+1} - \partial \beta_{1,j+1} \bar{Q}_1 - \partial \gamma_{j+1,1} \bar{J}_1 - \frac{1}{2} \text{tr}([n_j^u, A_\lambda] \bar{\partial} X)
\end{aligned}$$

$$\begin{aligned}
\chi_{j1} &= \det G^{-1} \det \begin{pmatrix} -\frac{1}{2}\text{tr}([n_j^d, A_\lambda]\partial X) & g_{11} \\ -\frac{1}{2}\text{tr}([n_j^d, A_\lambda]\bar{\partial} X) & g_{12} \end{pmatrix} \\
\chi_{j2} &= \det G^{-1} \det \begin{pmatrix} g_{12} & -\frac{1}{2}\text{tr}([n_j^d, A_\lambda]\partial X) \\ \bar{g}_{11} & -\frac{1}{2}\text{tr}([n_j^d, A_\lambda]\bar{\partial} X) \end{pmatrix} \\
(\partial^2 X)_k^v &= \gamma_{1,1}^{-1} \left(\partial Q_{j+1} + \frac{1}{2}\text{tr}(\partial X[v_{j+1}, A_\lambda]) - \alpha_{1,1}Q_{j+1} - \alpha_{1,2}\bar{Q}_{j+1} \right) \\
(\partial^2 X)_k^u &= \gamma_{1,1}^{-1} \left(\partial J_{j+1} + \frac{1}{2}\text{tr}(\partial X[u_{j+1}, A_\lambda]) - \alpha_{1,1}J_{j+1} - \alpha_{1,2}\bar{J}_{j+1} \right) \\
(\partial^2 X)_k^d &= -i\text{tr}(\partial^2 X \Phi E_{k+1,k+1} \Phi^{-1}) \\
(\partial \bar{\partial} X)_k^v &= \gamma_{1,1}^{-1} \left(\partial \bar{Q}_{j+1} + \frac{1}{2}\text{tr}(\bar{\partial} X[v_{j+1}, A_\lambda]) - \alpha_{2,1}Q_{j+1} - \alpha_{2,2}\bar{Q}_{j+1} \right) \\
(\partial \bar{\partial} X)_k^u &= \gamma_{1,1}^{-1} \left(\partial \bar{J}_{j+1} + \frac{1}{2}\text{tr}(\bar{\partial} X[u_{j+1}, A_\lambda]) - \alpha_{2,1}J_{j+1} - \alpha_{2,2}\bar{J}_{j+1} \right) \\
(\partial \bar{\partial} X)_k^d &= -i\text{tr}(\partial \bar{\partial} X \Phi E_{k+1,k+1} \Phi^{-1}) \\
(\partial n_j^v)_k^v &= \gamma_{1,1}^{-1} \left(-\frac{1}{2}\text{tr}([n_j^v, A_\lambda]v_{k+1}) + \partial \gamma_{1,1} \delta_{kj} - \nu_{j1}Q_{k+1} - \nu_{j2}\bar{Q}_{k+1} \right) \\
(\partial n_j^v)_k^u &= \gamma_{1,1}^{-1} \left(-\frac{1}{2}\text{tr}([n_j^v, A_\lambda]u_{k+1}) - \nu_{j1}J_{k+1} - \nu_{j2}\bar{J}_{k+1} \right) \\
(\partial n_j^v)_k^d &= -i\text{tr}([n_j^v, A_\lambda]\Phi E_{k+1,k+1} \Phi^{-1}) \\
(\partial n_j^u)_k^v &= \gamma_{1,1}^{-1} \left(-\frac{1}{2}\text{tr}([n_j^u, A_\lambda]v_{k+1}) - \mu_{j1}Q_{k+1} - \mu_{j2}\bar{Q}_{k+1} \right) \\
(\partial n_j^u)_k^u &= \gamma_{1,1}^{-1} \left(-\frac{1}{2}\text{tr}([n_j^u, A_\lambda]u_{k+1}) + \partial \gamma_{1,1} \delta_{jk} - \mu_{j1}J_{k+1} - \mu_{j2}\bar{J}_{k+1} \right) \\
(\partial n_j^u)_k^d &= -i\text{tr}([n_j^u, A_\lambda]\Phi E_{k+1,k+1} \Phi^{-1}) \\
(\partial n_j^d)_k^v &= \gamma_{1,1}^{-1} \left(-\frac{1}{2}\text{tr}([n_j^d, A_\lambda]v_{k+1}) - \chi_{j1}Q_{k+1} - \chi_{j2}\bar{Q}_{k+1} \right) \\
(\partial n_j^d)_k^u &= \gamma_{1,1}^{-1} \left(-\frac{1}{2}\text{tr}([n_j^d, A_\lambda]u_{k+1}) - \chi_{j1}J_{k+1} - \chi_{j2}\bar{J}_{k+1} \right) \\
(\partial n_j^d)_k^d &= -i\text{tr}([n_j^d, A_\lambda]\Phi E_{k+1,k+1} \Phi^{-1})
\end{aligned}$$

The other equations can be obtained by taking hermitian conjugation and by using the fact that X , n_j^v , n_j^u and n_j^d all belong to $\mathfrak{su}(N+1)$.

The second fundamental form (4.9) can now be read off from the Gauss-Weingarten equation

$$(\partial^2 X)_\perp = \sum_{k=1}^{\frac{N(N-1)}{2}} ((\partial^2 X)_k^v n_k^v + (\partial^2 X)_k^u n_k^u)$$

$$\begin{aligned}
& + \sum_{k=1}^N (\partial^2 X)_k^d n_k^d \\
(\partial \bar{\partial} X)_\perp &= \sum_{k=1}^{\frac{N(N-1)}{2}} ((\partial \bar{\partial} X)_k^v n_k^v + (\partial \bar{\partial} X)_k^u n_k^u) \\
& + \sum_{k=1}^N (\partial \bar{\partial} X)_k^d n_k^d \\
(\partial^2 X)_k^v &= \gamma_{1,1}^{-1} \left(\partial Q_{j+1} + \frac{1}{2} \text{tr}(\partial X[v_{j+1}, A_\lambda]) - \alpha_{1,1} Q_{j+1} - \alpha_{1,2} \bar{Q}_{j+1} \right) \\
(\partial^2 X)_k^u &= \gamma_{1,1}^{-1} \left(\partial J_{j+1} + \frac{1}{2} \text{tr}(\partial X[u_{j+1}, A_\lambda]) - \alpha_{1,1} J_{j+1} - \alpha_{1,2} \bar{J}_{j+1} \right) \\
(\partial^2 X)_k^d &= -i \text{tr}(\partial^2 X \Phi E_{k+1,k+1} \Phi^{-1}) \\
(\partial \bar{\partial} X)_k^v &= \gamma_{1,1}^{-1} \left(\partial \bar{Q}_{j+1} + \frac{1}{2} \text{tr}(\bar{\partial} X[v_{j+1}, A_\lambda]) - \alpha_{2,1} Q_{j+1} - \alpha_{2,2} \bar{Q}_{j+1} \right) \\
(\partial \bar{\partial} X)_k^u &= \gamma_{1,1}^{-1} \left(\partial \bar{J}_{j+1} + \frac{1}{2} \text{tr}(\bar{\partial} X[u_{j+1}, A_\lambda]) - \alpha_{2,1} J_{j+1} - \alpha_{2,2} \bar{J}_{j+1} \right) \\
(\partial \bar{\partial} X)_k^d &= -i \text{tr}(\partial \bar{\partial} X \Phi E_{k+1,k+1} \Phi^{-1}) \\
\alpha_{1,1} &= (\det G)^{-1} \left(\frac{1}{2} \partial g_{11} \bar{g}_{11} - \partial g_{12} g_{12} + \frac{1}{2} \bar{\partial} g_{11} g_{12} \right) \\
\alpha_{1,2} &= (\det G)^{-1} \left(g_{11} \partial g_{12} - \frac{1}{2} g_{11} \bar{\partial} g_{11} - \frac{1}{2} g_{12} \partial g_{11} \right) \\
\alpha_{2,1} &= (2 \det G)^{-1} (\bar{\partial} g_{11} \bar{g}_{11} - \partial \bar{g}_{11} g_{12}) \\
\alpha_{2,2} &= (2 \det G)^{-1} (g_{11} \partial \bar{g}_{11} - \bar{\partial} g_{11} g_{12}) \\
\det G &= g_{11} g_{22} - g_{12}^2 = \left| \sum_{k,l=1}^{N+1} (y_k - y_l)^2 (\Xi_1)_{kl} (\Xi_1)_{lk} \right|^2 \\
&- \left(\sum_{k,l=1}^{N+1} (y_k - y_l)^2 (\Xi_1)_{kl} (\Xi_2)_{lk} \right)^2
\end{aligned}$$

The mean curvature and the Gaussian curvature are then

$$\begin{aligned}
H &= (\det G)^{-1} \left(g_{11} (\bar{\partial}^2 X)^\perp - 2g_{12} (\partial \bar{\partial} X) + g_{22} (\partial^2 X)^\perp \right) \\
K &= (2 \det G)^{-1} \left(\partial \left(\frac{1}{\det G} (g_{12} \bar{\partial} \ln g_{11} - \partial g_{22}) \right) \right. \\
&\quad \left. + \bar{\partial} \left(\frac{1}{\det G} (2\partial g_{12} - \bar{\partial} g_{11} - g_{12} \partial \ln g_{11}) \right) \right)
\end{aligned}$$

and the Willmore functional is

$$W = \int_{\Omega} |H|^2 \sqrt{\det G} d\xi d\bar{\xi}$$

6 Explicit construction in the $\mathfrak{su}(2)$ case

In general, it is difficult to obtain formula for the matrix entries L_m with the Drinfeld-Sokolov iteration process [12]. In the 2×2 case, however, one can derive general formula for the matrix entries of L_m explicitly in terms of the solution u_0 by using the compatibility conditions (4.1)

$$\begin{aligned} \partial L_m &= [L_m, A_\lambda] \\ \bar{\partial} L_m &= [L_m, B_\lambda] \end{aligned}$$

In this case, the matrices $A(\lambda)$ and $B(\lambda)$ take the form

$$A_\lambda = \begin{pmatrix} \partial u_0 & \lambda e^{2u_0} \\ e^{-2u_0} & -\partial u_0 \end{pmatrix}, \quad B_\lambda = -\rho(A_\lambda)$$

Let the entries of L_m be the following

$$L_m = \begin{pmatrix} P(\lambda) & Q(\lambda) \\ R(\lambda) & -P(\lambda) \end{pmatrix}$$

which are polynomials in λ

$$\begin{aligned} P(z) &= \sum_{i=0}^m P_i \lambda^i \\ Q(z) &= \sum_{i=0}^m Q_i \lambda^i \\ R(z) &= \sum_{i=0}^m R_i \lambda^i \end{aligned}$$

By comparing the coefficients of λ in

$$\partial L_m = [L_m, A_\lambda]$$

we find that P , Q and R satisfy

$$\begin{aligned} \partial P(\lambda) &= Q(\lambda) e^{-2u_0} - \lambda R(\lambda) e^{2u_0} \\ \partial Q(\lambda) &= 2(P(\lambda) \lambda e^{2u_0} - Q(\lambda) \partial u_0) \\ \partial R(\lambda) &= 2(-P(\lambda) e^{-2u_0} + R(\lambda) \partial u_0) \end{aligned}$$

We first write $Q = e^{-2u_0}Q'$ and $R = e^{2u_0}R'$. This will simplify the last two equations

$$\begin{aligned}\partial P(\lambda) &= Q'(\lambda)e^{-4u_0} - \lambda R'(\lambda)e^{4u_0} \\ \partial Q'(\lambda) &= 2e^{4u_0}P(\lambda)\lambda \\ \partial R'(\lambda) &= -2e^{-4u_0}P(\lambda)\end{aligned}\tag{6.1}$$

By eliminating P from the above equations, we can express $\partial Q'$ in terms of R' , then by eliminating R' from the last two equations, we obtain

$$\lambda(16\partial u_0 + 4\partial)R' = (\partial^3 + 12\partial u_0\partial^2 + (32(\partial u_0)^2 + 4\partial^2 u_0)\partial)R'$$

We now do another change of variable $R' = e^{-4u_0}R''$ to obtain

$$4\lambda\partial R'' = (\partial^3 + (-8\partial^2 u_0 - 16(\partial u_0)^2)\partial + (-16\partial^2 u_0\partial u_0 - 4\partial^3 u_0))R''$$

We now denote \mathcal{L}_n by the following recursion operator

$$\begin{aligned}\partial\mathcal{L}_n[u_0] &= \frac{1}{4}\left(\partial^3 + (-8\partial^2 u_0 - 16(\partial u_0)^2)\partial\right. \\ &\quad \left.+ (-16\partial^2 u_0\partial u_0 - 4\partial^3 u_0)\right)\mathcal{L}_{n-1}[u_0] \\ &= \frac{1}{4}\left(\partial^3 - 8(\partial^2 e^{2u_0})\partial - 4(\partial^3 e^{2u_0})\right)\mathcal{L}_{n-1}[u_0] \\ \mathcal{L}_0[u_0] &= 1, \quad \mathcal{L}_k[u_0] = 0, \quad k < 0\end{aligned}\tag{6.2}$$

for example, the first few terms are

$$\begin{aligned}\mathcal{L}_0[u_0] &= 1 \\ \mathcal{L}_1[u_0] &= -\partial^2 e^{2u_0} \\ \mathcal{L}_2[u_0] &= -\frac{1}{16}\partial^4 e^{2u_0} + \frac{9}{8}\partial^2 e^{2u_0}\end{aligned}$$

We can now express P , Q and R in terms of \mathcal{L}_n

$$\begin{aligned}P &= \sum_{i=0}^m \left(2\partial u_0 \mathcal{L}_{m-i-1}[u_0] - \frac{1}{2}\partial \mathcal{L}_{m-i-1}[u_0]\right) \lambda^i \\ Q &= \sum_{i=0}^m \mathcal{L}_{m-i}[-u_0] e^{2u_0} \lambda^i \\ R &= \sum_{i=0}^m \mathcal{L}_{m-i-1}[u_0] e^{-2u_0} \lambda^i\end{aligned}\tag{6.3}$$

We can obtain the coefficients of $\rho(L_m)$ by

$$\tilde{P} = \overline{P}, \quad \tilde{Q} = \overline{R}, \quad \tilde{R} = \overline{Q}, \quad |\lambda| = 1$$

where \tilde{P} , \tilde{Q} , \tilde{R} are the entries of the matrix $\rho(L_m)$.

For $m = 2$, the entries are

$$\begin{aligned} P &= 2\partial u_0 \lambda + \frac{1}{2}\partial^3 u_0 - 4(\partial u_0)^3 \\ Q &= e^{2u_0} \left(\lambda^2 - \partial^2 e^{-2u_0} \lambda - \frac{1}{16}\partial^4 e^{-2u_0} + \frac{9}{8}\partial^2 e^{-2u_0} \right) \\ R &= e^{-2u_0} (\lambda - \partial^2 e^{2u_0}) \end{aligned}$$

We can now substitute the theta function solution (3.8) for u_0 (as defined in proposition 3.7) and express the immersion

$$X = i(L_m + \rho(L_m))$$

in terms of theta function. In the case of $m = 2$, this gives the matrix entries as follows

$$\begin{aligned} P &= \partial \log \Theta \lambda + \frac{1}{4}\partial^3 \log \Theta - \frac{1}{2}(\partial \log \Theta)^3 \\ Q &= \Theta \left(\lambda^2 - \partial^2 \Theta^{-1} \lambda - \frac{1}{16}\partial^4 \Theta^{-1} + \frac{9}{8}\partial^2 \Theta^{-1} \right) \\ R &= \Theta^{-1} (\lambda - \partial^2 \Theta) \end{aligned} \tag{6.4}$$

where Θ is the theta function expression on the right hand side of (3.8).

The components of the induced metric of the surface are the following

$$\begin{aligned} g_{11} &= -\frac{1}{2}\text{tr}(\partial X \partial X) = -(\partial P + \partial \bar{P})^2 - \frac{1}{2}(\partial(Q + \bar{R}))(\partial(R + \bar{Q})) \\ g_{12} &= -\frac{1}{2}\text{tr}(\partial X \bar{\partial} X) = -|(\partial P + \partial \bar{P})|^2 - \frac{1}{2}(\partial(Q + \tilde{Q}))(\bar{\partial}(R + \tilde{R})) \\ &\quad - \frac{1}{2}(\partial(R + \tilde{R}))(\bar{\partial}(Q + \tilde{Q})) \\ &= -|(\partial P + \partial \bar{P})|^2 - \frac{1}{2}(|\partial Q|^2 + |\bar{\partial} Q|^2 + |\partial R|^2 + |\bar{\partial} R|^2) \\ &\quad - \frac{1}{2}(\mathbf{Re} \partial R \bar{\partial} Q + \mathbf{Re} \partial Q \bar{\partial} R) \\ g_{22} &= \bar{g}_{11} \end{aligned}$$

In the $\mathfrak{su}(2)$ case, since we have

$$\text{tr}(X[A, X]) = \text{tr}(X[B, X]) = 0$$

we see that $\sqrt{-2\text{tr}(X^2)^{-1}}X$ is a unit normal vector to the surface. We have the following induced metric on the surface \mathcal{F}

$$\begin{aligned} g_{11} &= -\frac{1}{2}\text{tr}(X \partial^2 X) = -\frac{1}{2}\text{tr}(X[[X, A], A]) = -\frac{1}{2}\text{tr}([X, A]^2) \\ g_{12} &= -\frac{1}{2}\text{tr}(X \partial \bar{\partial} X) = -\frac{1}{2}\text{tr}(X[[X, B], A]) = \frac{1}{2}\text{tr}([B, X][X, A]) \\ g_{22} &= -\frac{1}{2}\text{tr}(X \bar{\partial}^2 X) = -\frac{1}{2}\text{tr}(X[[X, B], B]) = -\frac{1}{2}\text{tr}([B, X]^2) \end{aligned} \tag{6.5}$$

We can now use these relations to compute the second fundamental form and obtain the proportionality of the first and second fundamental forms

$$II = \sqrt{-2\text{tr}(X^2)^{-1}}(g_{11}d\xi d\xi + 2g_{12}d\xi d\bar{\xi} + g_{22}d\bar{\xi} d\bar{\xi}) = (\sqrt{-2\text{tr}(X^2)^{-1}})I$$

Hence the mean curvature H and Gauss curvature K of the surface \mathcal{F} are constants

$$\begin{aligned} H &= -(\det(G)\text{tr}(X^2))^{-1}(2g_{11}g_{22} + 2g_{12}^2) = -2\sqrt{\frac{2}{\text{tr}(X^2)}} \\ K &= 4\frac{\det(G)\text{tr}(X^2)^{-2}}{\det(G)} = -2\text{tr}(X^2)^{-1} \\ \det G &= -\frac{1}{2}(\text{tr}([B, X][X, A])^2 + \text{tr}([B, X]^2)\text{tr}([X, A]^2)) \end{aligned}$$

This is not surprising as $\text{tr}(X^2)$ is a constant and hence the surfaces are all mapped into the sphere S^2 of radius $\text{tr}(X^2)$. This means that different Toda lattice data u_0 correspond to different parametrizations of the same surface which is a subset of a sphere in \mathbb{R}^3 .

Finally we may determine formally a moving frame of the surface \mathcal{F} and write the structural equations, namely the Gauss-Weingarten equations and the Gauss-Codazzi equations. Let us assume that u_0 is a solution of the Toda lattice equations (2.1) such that $\det G \neq 0$ in some open neighborhood of a regular point $(\xi, \bar{\xi})$ in $\Omega \in \mathbb{C}$. Suppose also that the surface \mathcal{F} given by (4.7) is associated with these structural equations. The surface \mathcal{F} can be described by the moving frame

$$\eta = (\eta_1 = \partial X, \eta_2 = \bar{\partial} X, \eta_3, \dots, \eta_{N^2-1})^T$$

where $\eta_3, \dots, \eta_{N^2-1}$ are real normal vectors to the surface \mathcal{F} . The components of the moving frame η can be written in terms of $N \times N$ skew-hermitian matrices satisfying the following normalization conditions

$$\begin{aligned} (\partial X, \partial X) &= g_{11}, \quad (\partial X, \bar{\partial} X) = g_{12}, \quad (\bar{\partial} X, \bar{\partial} X) = g_{22} \\ (\partial X, \eta_k) &= (\bar{\partial} X, \eta_k) = 0, \quad (\eta_j, \eta_k) = (\eta_j, \eta_j)\delta_{jk}, \quad j, k = 3, \dots, N^2 - 1 \end{aligned}$$

In the $\mathfrak{su}(2)$ case, such a moving frame on \mathcal{F} consists of the the vectors ∂X , $\bar{\partial} X$ and X .

Let the frame $\eta = (\partial X, \bar{\partial} X, X)^T$, we have the following Gauss-Weingarten equations

$$\begin{aligned} \partial \eta &= M_1 \eta, \quad \bar{\partial} \eta = M_2 \eta \\ M_1 &= \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & 2g_{11} \\ \alpha_{2,1} & \alpha_{2,2} & 2g_{12} \\ 1 & 0 & 0 \end{pmatrix} \\ M_2 &= \begin{pmatrix} \alpha_{2,1} & \alpha_{2,2} & 2g_{12} \\ \bar{\alpha}_{1,2} & \bar{\alpha}_{1,1} & 2\bar{g}_{11} \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

where $\alpha_{i,j}$ are given by the following

$$\begin{aligned}\alpha_{1,1} &= (\det G)^{-1} \left(\frac{1}{2} \partial g_{11} \bar{g}_{11} - \partial g_{12} g_{12} + \frac{1}{2} \bar{\partial} g_{11} g_{12} \right) \\ \alpha_{1,2} &= (\det G)^{-1} \left(g_{11} \partial g_{12} - \frac{1}{2} g_{11} \bar{\partial} g_{11} - \frac{1}{2} g_{12} \partial g_{11} \right) \\ \alpha_{2,1} &= (2 \det G)^{-1} (\bar{\partial} g_{11} \bar{g}_{11} - \partial \bar{g}_{11} g_{12}) \\ \alpha_{2,2} &= (2 \det G)^{-1} (g_{11} \partial \bar{g}_{11} - \bar{\partial} g_{11} g_{12})\end{aligned}$$

which are the usual Christoffel symbols of second kind.

The coefficients $\alpha_{1,1}$ and $\alpha_{1,2}$ can be computed by solving the following linear equations

$$\begin{aligned}\text{tr}((\partial^2 X - \alpha_{1,1} \partial X - \alpha_{1,2} \bar{\partial} X) \partial X) &= 0 \\ \text{tr}((\partial^2 X - \alpha_{1,1} \partial X - \alpha_{1,2} \bar{\partial} X) \bar{\partial} X) &= 0\end{aligned}\tag{6.6}$$

Since from (6.6), we have

$$\partial^2 X = \alpha_{1,1} \partial X + \alpha_{1,2} \bar{\partial} X + \gamma X$$

for some γ , a solution $\alpha_{1,1}$ and $\alpha_{1,2}$ to the (6.6) would give the ∂X and $\bar{\partial} X$ components of $\partial^2 X$. To determine γ , note that since X is orthogonal to the plane spanned by ∂X and $\bar{\partial} X$, the coefficient γ is just the inner product between $\partial^2 X$ and X

$$\gamma = \frac{1}{2} \text{tr}(\partial^2 X X) = g_{11}$$

which follows from (6.5). The other coefficients can be computed similarly.

7 Conclusions

In this paper we have studied the $\mathfrak{su}(N)$ algebraic description of surfaces obtained from theta function solutions of the 2D Toda lattice. We have derived the generalized Weierstrass formula for immersion which is expressed in terms of the Lax operators. It allows us to construct surfaces immersed in $\mathfrak{su}(N)$ algebra and show that all these surfaces are mapped into hyperspheres in \mathbb{R}^{N^2-1} . It has been proved to be effective, since we were able to recover easily the known results associated to the $\mathfrak{su}(2)$.

The structural equations of the 2-dimensional surfaces, their moving frames, the first and second fundamental forms, the Gaussian curvature, the mean curvature vector have been expressed in terms of the Baker function of the Toda lattice. The implementation of these theoretical results have been applied to the $\mathfrak{su}(2)$ case, leading to the negative constant Gaussian curvature, which means that these surfaces are subsets of a sphere in \mathbb{R}^3 .

The approach presented here is limited to theta function solutions of the 2D Toda lattice. The question arises as to whether this approach can be extended to integrable systems (for example to the \mathbb{CP}^N models in (2+1)-dimensions or non-Abelian gauge theories) and if these models can lead to the construction of other classes of submanifolds immersed in multi-dimensional spaces. Other aspects of the method worth investigating are the completeness of solutions, the compactness of surfaces and the stability of the resulting surfaces.

The geometrical analysis of surfaces and their deformations under various types of dynamics have generated a great deal of interest and activities in several mathematical and physical areas of research as well. In particular, it is worth noting that we know in certain cases the analytic description of surfaces in a physical system for which analytic models are not yet fully developed. However, by using our approach we can in some cases select an appropriate Toda lattice model corresponding to the given Weierstrass representation and characterize the classes of equations describing the physical phenomena under investigation. This approach was attempted successfully for the Weierstrass representation associated to sine-Gordon model, for surfaces with negative Gaussian curvature in three-dimensional Euclidean space [4], [35], but not to our knowledge for multi-dimensional spaces.

References

- [1] D. AMIT. *Field theory, the renormalization group and critical phenomena*. Mc-Graw Hill, N. Y., 1978.
- [2] A. BOBENKO. Surface in terms of 2×2 matrices, in: *Harmonic maps and integrable systems*. Ed. A. Fordy and J. C. Wood, Vieweg, Braunschweig, (1994).
- [3] A. BOBENKO and U. EITNER. *Painlevé equations in differential geometry of surfaces*. Lect. Notes. Math. 1753, Springer, Berlin 2000.
- [4] K. GROSSE-BRAUCKMAN and K. POLTHIER. Constant mean curvature surface derived from Delaunay's and Wente's examples, in: *Visualization and Mathematics: Experiments, simulations and environments*. Ed. H. C. Hege and K. Polthier, Springer-Verlag, Berlin, (1997).
- [5] P. BUDINICH and M. RIGOLI. Cartan spinors, minimal surfaces and strings. *il nuovo cimento, B*, 102: N. 6, (1988).
- [6] J. CHAVOLIN, J. F. JOANNY, J. ZINN-JUSTIN. *Liquids of interfaces*. Elsevier, Amsterdam, 1989.
- [7] F. DAVID, P. GINSPIRG and Y. ZINN-JUSTIN (Editors). *Fluctuating geometries in statistical mechanics and field theory*. Elsevier, Amsterdam, 1996.

- [8] A. DAVIDOV. *Solitons in molecular systems*. Klumer, N.Y., 1991.
- [9] G. DE MATTEIS. Group analysis of the shape equation. *Proceedings of conf. on nonlinear physics: Theory and experiment II*, Eds. H. Ablowitz et al, World scientific, Singapore, 227-249, (2002).
- [10] M. P. DO CARMO, *Riemannian geometry, Mathematics: Theory and applications*. Birkhäuser, Boston 1992.
- [11] J. DORFMEISTER, F. PEDIT and H. WU. Weierstrass type representation of harmonic maps into symmetric spaces. *Comm. Anal. Geom.* 6: no. 4, 633-668, (1998).
- [12] V. G. DRINFELD and V. V. SOKOLOV. Equations of Korteweg-de Vries type and simple Lie algebras. *Soviet Math. Dokl.*, 23: 3, 457-62, (1981).
- [13] B. DUBROVIN. Theta functions and non-linear equations. *Russian Math. Surveys*, 36: 11-92, (1981).
- [14] A. ENNEPER. Analytisch-geometrische Untersuchungen. *Göttingen Nachr.* 12: 258-277, 421-443, (1868).
- [15] T. FRIEDRICH. On the spinor representation of surfaces in Euclidean 3-spaces. *J. Geom. and Phys.* 28: 143-157, (1998).
- [16] M. A. GUEST, *Harmonic maps, loop groups and integrable systems*. Cambridge university press, Cambridge 1997.
- [17] F. HELEIN, *Constant mean curvature surfaces, harmonic maps and integrable systems*. Lect. Math., Birkhäuser, Boston 2001.
- [18] F. HELEIN, *Harmonic maps, conservation laws and moving frames*. Cambridge university press, Cambridge 2002.
- [19] S. HELGASON, *Differential geometry, Lie groups and symmetric space*. Academic press, New York, 1978.
- [20] T. IOANNIDOU, G. PAPADOPOULOS and P. M. SUTCLIFFE. Non-BPS Dirac-Born-Infeld solitons. *J. High Energy Physics* no. 9, paper 16: 14, (1999).
- [21] N. N. JANENKO and B. L. ROZDESTUENSKII. *Systems of quasilinear equations and their applications to gas dynamics*. Translations of mathematical monographs vol. 55, AMS, 1983.
- [22] K. KENMOTSU, *Surfaces with constant mean curvatures*, AMS, Providence, R. I. 2003.
- [23] S. KOBAYASHI and K. NOMIZU, *Foundation of differential geometry*, John Wiley, New York. vol. 1 and 2. 1963.

- [24] B. KONOPELCHENKO and I. TAIMANOV. Constant mean curvature surfaces via integrable dynamical system. *J. Phys. A: Math. Gen* 29: 1261-1265, (1996).
- [25] G. LANDOLFFI. On the Canham-Helfrich membrane model. *J. Phys. A* 36: 4677-4715, (2003).
- [26] R. LIPOWSKI and E. SACKMAN (Editors). *Handbook of biological Physics, vol. 1, Structure and dynamics of membranes*, Elsevier, Amsterdam, 1995.
- [27] D. MUMFORD, *Tata lectures on theta vol. 1 and 2*. Birkhäuser, Boston. 1983-84.
- [28] P. NELSON, T. PIRAN and S. WEINBERG. *Statistical mechanics of membranes and surfaces*. World scientific, Singapore, 1992.
- [29] R. OSERMANN, *A survey of minimal surfaces*. Dover, New York. 1996.
- [30] Z. OU-YANG, J. LUI and Y. XIE. *Geometric methods in elastic theory of membranes in liquid crystal phase*, World scientific, Singapore, 1999.
- [31] F. PEDIT and U. PINKALL, Quaternionic analysis on Riemannian surfaces and differential geometry. *Doc. Math. Extra volume ICM*. 389-400, (1998).
- [32] U. PINKALL and STERLING, On the classification of constant mean curvature tori. *Ann. Math.* 130: 407-451, (1989).
- [33] U. PINKALL, Regular homotopy classes of immersed surfaces, *Topology* 24: 421-434, (1985).
- [34] J. POLCHINSKI. *String theory*. CUP, Cambridge, 1992.
- [35] C. ROGERS and W. K. SCHIEF. *Bäcklund and Darboux transformations. Geometry and modern applications in soliton theory*, Cambridge texts in applied mathematics, Cambridge university press, (2002).
- [36] S. A. SAFRAM. *Statistical thermodynamics of surfaces, interfaces and membranes*. Addison-Wesley, N. Y. 1994.
- [37] K. WEIERSTRASS. Fortsetzung der Untersuchung über die Minimalflächen, *Mathematische Werke*, Verlagsbuch-handlung, Hillesheim, 3: 219-248, (1866). G. DARBOUX, *Lecons sur systemes orthogonaux et les coordonnees curvilignes*. (Gauthier-Villas, Paris, 1910).
- [38] T. J. WILLMORE, *Riemannian geometry*, Clarendon press, Oxford, 1993.